BURNSIDE RING OF A FINITE GROUP

AHMET BERKAY KEBECI

Abstract. The aim of this project is to introduce the Burnside ring of a given finite group and to understand the ring structure of it using the mark homomorphism and Gluck Idempotent Formula. First, a detailed definition of the Burnside ring is given with its basic properties. Then the mark homomorphism is introduced to study the Burnside algebra using the ghost ring. Finally, a formula, called Gluck Idempotent Formula, is proven to calculate the basis elements of the Burnside algebra over $\mathbb{Q}$, which are the idempotents.

Introduction

Let G be a finite group. The Burnside ring of G is defined to be the Grothendieck ring of the free commutative monoid generated by isomorphism classes of finite G-sets. We will denote the Burnside ring of G by $\Omega(G)$. The addition and multiplication of $\Omega(G)$ are defined by disjoint unions and Cartesian products, respectively. As an abelian group $\Omega(G)$ is a free $\mathbb{Z}$-module generated by the isomorphism classes of finite transitive G-sets, which are in the one-to-one correspondence with the conjugacy classes of subgroups of G. By letting $Cl(G)$ denote the set of representatives of conjugacy classes of subgroups of G, all finite transitive G-sets are of the form G/H where G/H denotes the set of cosets of H in, G over $H \in Cl(G)$ (see [2, 15.1.2]).

Consider the symmetric group S_3 on three elements. We have

$$Cl(S_3) = \{(1), (C_2), (C_3), (S_3)\}$$

where C_n denotes the cyclic group of order n. Hence any transitive S_3-set is isomorphic to one of $S_3/1, S_3/C_2, S_3/C_3, S_3/S_3$. Therefore $\Omega(S_3)$ is freely generated by $[S_3/1], [S_3/C_2], [S_3/C_3], [S_3/S_3]$ as a $\mathbb{Z}$-module. For the multiplication in terms of basis elements in $\Omega(G)$, there is a nice formula called Mackey Product Formula stated and proven in Section 1.

In Section 2, we define the ghost ring consisting the super class functions from $Cl(G)$ to $\mathbb{Z}$ to define an injective ring homomorphism from the Burnside ring to the ghost ring. This homomorphism will be called the mark homomorphism. Thanks to these tools, we can construct a matrix called the table of marks which is a representation of the Burnside ring for a better understanding. Our first main theorem, Theorem 2.4, states that there is a short exact sequence of abelian groups where first map is the mark homomorphism. In light of this theorem, we construct an isomorphism between $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ and the ghost algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^*(G)$. Hence we can send the basis elements of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ into its ghost algebra and write them in terms of the primitive idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ with integer coefficients. Since we have a bijection, we can take the preimages of the primitive idempotents. Hence we may write the primitive idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ in terms of the transitive G-set basis elements of $\Omega(G)$ with rational coefficients.

In [1], Gluck has proven an idempotent formula which gives the coefficients discussed in Section 2. In Section 3, we give a proof of this formula by using combinatorial techniques including Mobius inversion for partially ordered sets. Because of that, we discuss posets in the beginning of this section. Gluck idempotent formula, Theorem 3.4, is the second main theorem of this paper.

In the first two sections notations, definitions and results mostly followed from [2], whereas those in Section 3 are mostly followed from [1].

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1. Burnside Rings

Aim of this chapter is to introduce the Burnside ring of a finite group G and discuss the main properties of it. Let F be the free Abelian group

$$\langle (X) : (X) \text{ is an isomorphism class of the } G\text{-set } X \rangle.$$

We define multiplication in F as $(X)(Y) = (X \times Y)$, therefore F becomes a commutative ring. Let F_0 be the additive subgroup of F generated by all $(X \sqcup Y) - (X) - (Y)$, then F_0 is an ideal of F. Now we are ready to define the Burnside ring of G.

Definition 1.1. Let F and F_0 be as above for a finite group G. $\Omega(G) = F/F_0$ is called the *Burnside ring* of G.

Therefore the elements of $\Omega(G)$ are of the form $[X] = (X) + F_0$. Addition in $\Omega(G)$ is given by

$$[X] + [Y] = [X \sqcup Y]$$

and the multiplication is determined by

$$[X][Y] = [X \times Y]$$

for every G-sets X and Y. Thus $\Omega(G)$ is a commutative ring with identity $[G/G]$ and zero element $[\emptyset]$.

Krull-Schmidt Theorem for G-sets (see [2, 15.1.8]) states that for any G-set X,

$$X \cong \bigsqcup_{i=1}^{n} \lambda_i (G/H_i)$$

where $H_1, H_2, ..., H_n$ are representatives of all G-conjugacy classes of subgroups of G and

$$\lambda_i = \frac{|\{x \in X : G_x \text{ is } G\text{-conjugate to } H_i\}|}{[G : H_i]}.$$

Using this fact we will prove the following result to understand elements of the Burnside ring .

Theorem 1.2. *Let X and Y be G-sets. Then $[X] = [Y]$ in $\Omega(G)$ if and only if $X \cong Y$.*

Proof. It is trivial that $X \cong Y$ implies $[X] = [Y]$. Assume that $[X] = [Y]$ i.e. $(X) + F_0 = (Y) + F_0$ so $(X) - (Y) \in F_0$. Then we may write

$$(X) - (Y) = \sum_i \{(A_i \sqcup A_i') - (A_i) - (A_i')\} - \sum_j \{(B_j \sqcup B_j') - (B_j) - (B_j')\}$$

and therefore

$$(X) + \sum_i (A_i) + \sum_i (A_i') + \sum_j (B_j \sqcup B_j') = (Y) + \sum_j (B_j) + \sum_j (B_j') + \sum_i (A_i \sqcup A_i').$$

If we define $A = \bigsqcup_i A_i$, $A' = \bigsqcup_i A_i'$, $B = \bigsqcup_i B_i$ and $B' = \bigsqcup_i B_i'$ then the above equation implies that

$$X \sqcup A \sqcup A' \sqcup (B \sqcup B') \cong Y \sqcup B \sqcup B' \sqcup (A \sqcup A')$$

so by letting $S = A \sqcup A' \sqcup B \sqcup B'$, we have

$$X \sqcup S \cong Y \sqcup S.$$

Hence by Krull-Schmidt Theorem for G-sets (see [2, 15.1.8]), $X \cong Y$. □

Theorem 1.3. *Let $H_1, H_2, ..., H_n$ be representatives of all G-conjugacy classes of subgroups of G. Given a G-set X, define $X_i = \{x \in X : G_x$ is G-conjugate to $H_i\}$, for every $1 \leq i \leq n$. Then,*

$$[X] = \sum_{i=1}^{n} \lambda_i [G/H_i]$$

for uniquely determined integers

$$\lambda_i = \frac{|X_i|}{[G : H_i]}.$$

Proof. By Krull-Schmidt Theorem for G-sets (see [2, 15.1.8]),

$$X \cong \bigsqcup_{i=1}^{n} \lambda_i (G/H_i).$$

The result follows from Theorem 1.2. □

The following result will provide a good perspective for Burnside rings.

Theorem 1.4. *Let $H_1, H_2, ..., H_n$ be representatives of all conjugacy classes of G. Then $\Omega(G)$ is a free $\mathbb{Z}$-module generated by*

$$[G/H_1], [G/H_2], ..., [G/H_n].$$

Proof. Let $[X] \in \Omega(G)$. By Theorem 1.3,

$$[X] = \sum_{i=1}^{n} \lambda_i [G/H_i]$$

for some uniquely determined integers λ_i. Hence $[G/H_i]$'s generate $\Omega(G)$ as a $\mathbb{Z}$-module. Assume

$$\sum_{i=1}^{n} \lambda_i [G/H_i] = 0$$

for some integers λ_i. Without loss of generality we may assume that $\lambda_i \geq 0$ when $i \in \{1, ..., r\}$ and $\lambda_i < 0$ otherwise for some integer $1 \leq r \leq$ n. Then,

$$\sum_{i=1}^{r} \lambda_i [G/H_i] = \sum_{i=r+1}^{n} -\lambda_i [G/H_i].$$

Therefore by the uniqueness of λ_i's in Theorem 1.3 we have $\lambda_i = 0$ for every $1 \leq i \leq n$ which concludes that $\Omega(G)$ is freely generated by $[G/H_i]$'s. □

Example 1.5. Consider S_3, the symmetric group of order 6 and D_8, the dihedral group of order 8.

(i) There are four conjugacy classes in S_3 which are isomorphic to $1, C_2, C_3$ and S_3. Hence we can write the Burnside ring of S_3 as

$$\Omega(S_3) = [S_3/S_3]\mathbb{Z} + [S_3/C_3]\mathbb{Z} + [S_3/C_2]\mathbb{Z} + [S_3/1]\mathbb{Z}.$$

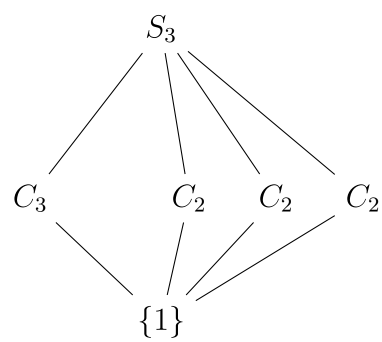

Figure 1. Subgroup lattice of S_3

(ii) D_8 has 10 subgroups and 8 conjugacy classes of subgroups.

$$Cl(D_8) = \left\{(1), (C_2^{(1)}), (C_2^{(2)}), (C_2^{(Z)}), (C_4), (V_4^{(1)}), (V_4^{(2)}), (D_8)\right\}.$$

Therefore,

$$\Omega(D_8) = [D_8/D_8]\mathbb{Z} + [D_8/V_4^{(1)}]\mathbb{Z} + [D_8/C_4]\mathbb{Z} + [D_8/V_4^{(2)}]\mathbb{Z} + [D_8/C_2^{(1)}]\mathbb{Z} + [D_8/C_2^{(Z)}]\mathbb{Z} + [D_8/C_2^{(2)}]\mathbb{Z} + [D_8/1]\mathbb{Z}.$$

Hence we have a basis for $\Omega(G)$. However we still do not have much information about the multiplicative structure of $\Omega(G)$. The next result will lead us to make the multiplication table of $\Omega(G)$.

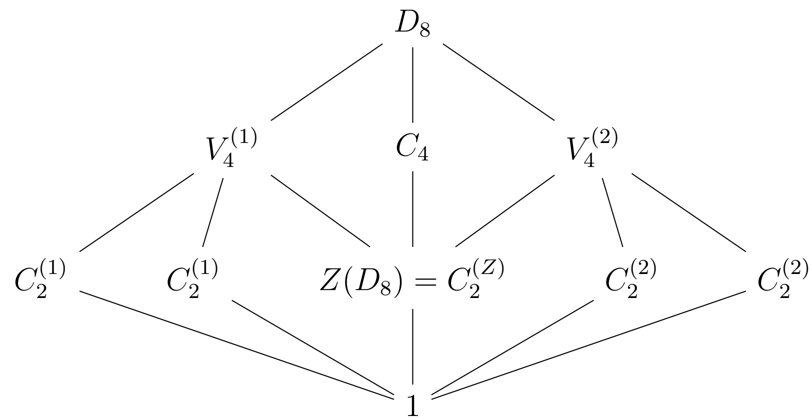

FIGURE 2. Subgroup lattice of D_8

Lemma 1.6 (Mackey Product Formula)**.** *For any subgroups H and K of G,*

$$[G/H][G/K] = \sum_{HgK\subseteq G} [G/(H \cap gKg^{-1})] \text{ in } \Omega(G)$$

where the notation indicates that g runs through the representatives of the $H\backslash G/K$ double cosets.

Proof. By Theorem 1.2, it suffices to prove that

$$(G/H) \times (G/K) \cong \bigsqcup_{HgK\in G} [G/(H \cap gKg^{-1})].$$

Let $X = (G/H) \times (G/K)$ and for every $g \in G$ define $\bar{g} = (H, gK) \in X$. Then

$$\begin{aligned} G_{\bar{g}} = \{g' \in G : g'\bar{g} = \bar{g}\} &= \{g' \in G : g'H = H, g'gK = gK\} \\ &= \left\{g' \in G : g' \in H, g^{-1}g'g \in K\right\} = H \cap gKg^{-1}. \end{aligned}$$

Hence by Orbit-Stabilizer Theorem (see [2, 15.1.1]), $Orb_G(\bar{g}) \cong G/(H \cap gKg^{-1})$.

Notice that $Orb_G(\bar{a}) = Orb_G(\bar{b})$ means that $\bar{a} = g\bar{b}$ for some $g \in G$. Therefore it is equivalent to $(H, aK) = (gH, gbK)$. Hence $Orb_G(\bar{a}) = Orb_G(\bar{b})$ if and only if $g \in H$ and $aK = gbK$ which is equivalent to $HaK = HgbK = HbK$. Hence all $Orb_G(\bar{g})$ are disjoint. Moreover any G-orbit of X has an element of the form $\bar{g}$ for some $g \in G$. Thus

$$X = \bigsqcup_{HgK\in G} Orb_G(\bar{g}) \cong \bigsqcup_{HgK\in G} [G/(H \cap gKg^{-1})].$$

□

Example 1.7. (i) Consider S_3, taking $C_2 = \langle(1,2)\rangle$ as the representative of (C_2) call $1 = [S_3/S_3]$, $x = [S_3/C_3]$, $y = [S_3/C_2]$ and $z = [S_3/1]$. We will show that the multiplication table of the basis elements of $\Omega(S_3)$ is the following.

	1	x	y	z
1	1	x	y	z
x	x	$2x$	z	$2z$
y	y	z	$y+z$	$3z$
z	z	$2z$	$3z$	$6z$

TABLE 1. Multiplication table of $\Omega(S_3)$

First notice that

$$[S_3/S_3][S_3/H] = \sum_{S_3 gH \subseteq S_3} [S_3/(S_3 \cap gHg^{-1})] = \sum_{S_3 \subseteq S_3} [S_3/H] = [S_3/H]$$

and

$$[S_3/1][S_3/H] = \sum_{1gH \subseteq S_3} [S_3/(1 \cap gHg^{-1})] = \sum_{gh \subseteq S_3} [S_3/1] = [G:H] \cdot [S_3/1]$$

for every $H \leqslant G$. Therefore $xz = 2x$, $yz = 3y$ and $z^2 = 6z$. Moreover,

$$x^2 = [S_3/C_3][S_3/C_3] = \sum_{C_3 gC_3 \subseteq S_3} [S_3/(C_3 \cap gC_3g^{-1})] = \sum_{C_3 gC_3 \subseteq S_3} [S_3/C_3] = 2[S_3/C_3] = 2x,$$

$$xy = [S_3/C_3][S_3/C_2] = \sum_{C_3 gC_2 \subseteq S_3} [S_3/(C_3 \cap gC_2g^{-1})] = \sum_{C_3 gC_2 \subseteq S_3} [S_3/1] = [S_3/1] = z$$

and finally,

$$(*) \quad y^2 = [S_3/C_2][S_3/C_2] = \sum_{C_2 gC_2 \subseteq S_3} [S_3/(C_2 \cap gC_2g^{-1})]$$

$$= [S_3/C_2 \cap 1C_21)] + [S_3/(C_2 \cap (1,2,3)C_2(1,3,2))] = [S_3/C_2] + [S_3/1] = y + z.$$

(ii) Consider C_6, the cyclic group of order 6. Put $1 = [C_6/C_6], x = [C_6/C_3], y = [S_6/C_2]$ and $z = [C_6/1]$. Then,

$$x^2 = \sum_{C_3 gC_3 \subseteq C_6} [C_6/C_3] = 2x,$$

$$xy = \sum_{C_3 gC_2 \subseteq C_6} [C_6/1] = z,$$

$$y^2 = \sum_{C_2 gC_2 \subseteq C_6} [C_6/C_2] = 3y.$$

Hence we have the following table.

	1	x	y	z
1	1	x	y	z
x	x	$2x$	z	$2z$
y	y	z	$3y$	$3z$
z	z	$2z$	$3z$	$6z$

TABLE 2. Multiplication table of $\Omega(C_6)$

Notice that in (*), we have found the double cosets as

$$\{1, (1,2)\}$$

and

$$\{(1,2,3), (1,3), (2,3), (1,3,2)\}$$

having cardinalities 2 and 4 whereas the G-sets corresponding them are of cardinalities 3 and 6, respectively. In what follows we investigate this relation and find a formula stated in Corollary 1.9.

Proposition 1.8. *Let H and K be subgroups of G and $g \in G$. Then the size of the double coset HgK is*

$$|HgK| = \frac{|H||K|}{|H \cap gKg^{-1}|}.$$

Proof. Consider the map $\phi : HgK \to HgKg^{-1}$ determined by $\phi(hgk) = hgkg^{-1}$ for every $h \in H, k \in K$. If $\phi(h_1gk_1)\phi(h_2gk_2)$, then $h_1gk_1g^{-1} = h_2gk_2g^{-1}$ so $h_1gk_1 = h_2gk_2$ which means that ϕ is injective. Given any $a \in HgKg^{-1}$ we have $\phi(ag) = a$, so ϕ is surjective. Therefore ϕ is a bijection.

Noting that gKg^{-1} is a subgroup of G,

$$|HgK| = |HgKg^{-1}| = \frac{|H||gKg^{-1}|}{|H \cap gkg^{-1}|} = \frac{|H||K|}{|H \cap gKg^{-1}|}.$$

□

Corollary 1.9. *Let H and K be subgroups of G, $g \in G$ and $X = (G/H) \times (G/K)$. Let $\bar{g} = (H, gK) \in X$. Then,*

$$|Orb_G(\bar{g})| = \frac{[G:H]}{|K|} \cdot |HgK|.$$

Proof. By Proposition 1.8,

$$|Orb_G(\bar{g})| = |G/H \cap gKg^{-1}| = \frac{[G:H]}{|K|} \cdot |HgK|.$$

□

2. The Mark Homomorphism

Let $Cl(G) = \{K_1, K_2, ..., K_n\}$ be the set of representatives of all conjugacy classes of a finite group G. We aim to continue exploring the Burnside ring of G. For this we will define another ring structure that is easier to write a proper basis. Since G/K_i's generate $\Omega(G)$, enumerating K_i's gives a characterization for all ring. Therefore the following concept we will define will be beneficial to understand $\Omega(G)$.

Definition 2.1. $\Omega^*(G) := \{f : Cl(G) \to \mathbb{Z} \mid f \text{ is a function}\}$ with for any $f_1, f_2 \in \Omega^*(G)$ and $K \in Cl(G)$,

$$(f_1 + f_2)(K) = f_1(K) + f_2(K)$$
$$(f_1 f_2)(K) = f_1(K) f_2(K)$$

is called the *ghost ring* of G.

Notice that $\Omega^*(G)$ is isomorphic to $\mathbb{Z}^{|Cl(G)|}$ as rings with a ring isomorphism

$$f \mapsto (f(K_1), f(K_2), ..., f(K_n)).$$

Definition 2.2. Given a G-set X and a subgroup H of G,

$$X^H = \{x \in X : hx = x, \forall h \in H\}$$

is called *H-invariant subset* of X.

The cardinality of X^H is usually referred as the number of H-fixed points of X and it will be an important concept. Consider the map

$$\varphi : \Omega(G) \to \Omega^*(G)$$

such that

$$X \mapsto f$$

where

$$f : Cl(G) \to \mathbb{Z} \text{ with } K \mapsto |X^K|$$

Let X and Y be G-sets and H be a subgroup of G, then one can say that

$$(X \sqcup Y)^H = X^H \sqcup Y^H$$
$$(X \times Y)^H = X^H \times Y^H$$

by Definition 2.2. Therefore φ is a ring homomorphism.

Definition 2.3. The homomorphism $\varphi : \Omega(G) \to \Omega^*(G)$ defined above is called the *mark homomorphism* of G.

If we see Ω^* as $\mathbb{Z}^{|Cl(G)|}$ then the mark homomorphism can be seen as

$$\varphi : \Omega(G) \to \mathbb{Z}^{|Cl(G)|}$$

where

$$X \mapsto (|X^{K_1}|, |X^{K_2}|, ..., |X^{K_n}|)$$

which provides the promised enumeration. Next, we will prove an important theorem about the mark homomorphism.

Theorem 2.4. *Let $Cl(G) = \{K_1, K_2, ..., K_n\}$ be the set of representatives of all conjugacy classes of a finite group G and define $r_i := [N_G(K_i) : K_i]$, also let $\Omega(G)$ be the Burnside ring of G and*

$$\Omega^*(G) = \{f : Cl(G) \to \mathbb{Z} \mid f \textit{ is a function}\}$$

be the ghost ring of G, then the sequence

$$0 \to \Omega(G) \xrightarrow{\varphi} \Omega^*(G) \xrightarrow{\psi} \bigoplus_{i=1}^{n} \mathbb{Z}/r_i\mathbb{Z} \to 0$$

is exact where

$$\varphi(X) = f_X \textit{ such that } f_X : Cl(G) \to \mathbb{Z} \textit{ with } f_X(K) = |X^K|$$

is the mark homomorphism and

$$\psi(f)_i = \sum_{gK_i \in N_G(K_i)/K_i} f(\langle g, K_i \rangle) \mod r_i.$$

We will prove Theorem 2.4 in four steps. First, we will show that the Burnside homomorphism φ is injective. After that we will show $\varphi(\Omega(G)) \subseteq \ker(\psi)$ and $\Omega^*(G)/\varphi(\Omega(G)) \subseteq \bigoplus_{i=1}^{n} \mathbb{Z}/r_i\mathbb{Z}$ as the second and the third steps. Finally we will deduce that ψ is surjective to conclude that $\varphi(\Omega(G)) = \ker(\psi)$ and finish the proof. However, it is more beneficial to introduce some tools, we will use in the proof, as separate lemmas.

Lemma 2.5 (Burnside's lemma). *Let G be a finite group and X be an arbitrary G-set. Let X/G be denote the set of G-orbits of X. Then we have*

$$|G| \cdot |X/G| = \sum_{g \in G} |X^{\langle g \rangle}|.$$

Proof. Notice that

$$\begin{aligned}
\sum_{g \in G} |X^{<g>}| &= \sum_{g \in G} |\{x \in X \ : \ g^n x = x, \forall n \in \mathbb{N}\}| \\
&= \sum_{g \in G} |\{x \in X \ : \ gx = x\}| \\
&= |\{(g,x) : gx = x, g \in G, x \in X\}| \\
&= \sum_{x \in X} |G_x|.
\end{aligned}$$

Then by Orbit-Stabilizer theorem,

$$\begin{aligned}
\sum_{g \in G} |X^{<g>}| = \sum_{x \in X} |G|/|Orb_G(x)| &= |G| \cdot \sum_{x \in X} 1/|Orb_G(x)| \\
&= |G| \cdot \sum_{A \in X/G} \sum_{x \in A} 1/|A| \\
&= |G| \cdot \sum_{A \in X/G} 1 = |G| \cdot |X/G|.
\end{aligned}$$

□

Lemma 2.6. *Let H and K be subgroups of a finite group G, then*

$$|(G/H)^K| = [N_G(H) : H] \cdot n(K, H)$$

where $n(K, H)$ is the number of G-conjugates of H containing K.

Proof.

$$\begin{aligned}
|(G/H)^K| = |\{gH \in G/H : KgH = gh, \forall k \in K\}| &= |\left\{gH \in G/H : g^{-1}kg \in H, \forall k \in K\right\}| \\
&= |\{gH \in G/H : K^g \subseteq H\}| = |\{gH \in G/H : K \subseteq {}^gH\}|.
\end{aligned}$$

Let ${}^{g_i}H$'s be the distinct G-conjugates of H containing K and let n_jH's be the cosets of H in $N_G(H)$ so $1 \le i \le n(K,H)$ and $1 \le j \le [N_G(H) : H]$. We will prove that g_in_jH's are the all distinct elements of $(G/H)^K$.

Since ${}^{g_in_j}H = {}^{g_i}H \supseteq K$, we have $g_in_jH \in (G/H)^K$ for all i, j. Assume $g_in_jH = g_an_bH$ for some $1 \le i, a \le n(K,H)$ and $1 \le j, b \le [N_G(H) : H]$. Then $g_i = g_an_bhn_j^{-1}$

for some $h \in H$ so ${}^{g_i}H = {}^{g_a n_b h n_j}H = {}^{g_a}H$. and therefore $i = a$ which leads that $n_j H = n_b H$ and $b = j$. Hence all $g_i n_j H$'s are distinct. Let $gH \in (G/H)^K$. Then $K \subseteq {}^gH$ which means ${}^gH = {}^{g_i}H$ for some i. Then $g_i^{-1}g \in N_G(H)$. Since every element should belong to a coset, $g_i^{-1}g \in n_j H$ for some j. Therefore $gH = g_i n_j H$ and we are done. □

Corollary 2.7. *Let H and K be subgroups of a finite group G, then*

(i) $|(G/K)^K| = [N_G(K) : K]$

(ii) $|(G/H)^K| \neq 0$ *if and only if K is contained in a G-conjugate of H.*

In light of the mark homomorphism we can write a matrix or table whose (K, H)-th entry is consisting of number of fixed points $|(G/H)^K|$ for $H, K \in Cl(G)$ which are usually called marks referring to the mark homomorphism. Using Lemma 2.6 we can write this matrix as the product of two matrices whose entries given by $n(K, H)$ and $[N_G(H) : H]$. Note that this matrix is upper triangular as what Corollary 2.7(ii) says.

Example 2.8. Consider S_3. Then the table of marks appears as follows.

	$S_3/1$	S_3/C_2	S_3/C_3	S_3/S_3
1	6	3	2	1
C_2	0	1	0	1
C_3	0	0	2	1
S_3	0	0	0	1

TABLE 3. Mark of tables of $\Omega(S_3)$

If we write it as a matrix we have

$$\begin{bmatrix} 6 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that $6, 1, 2, 1$ are $[N_G(H) : H]$'s and the other matrix in the right hand side is the matrix with entries given by $n(K, H)$'s.

Now we are ready to give proof of the theorem.

Proof of Theorem 2.4. Let X and Y be G-sets where $|X^K| = |Y^K|$, for any $K \in Cl(G)$. Then $X = \sum_{i=1}^{n} \alpha_i [G/K_i]$ and $Y = \sum_{i=1}^{n} \beta_i [G/K_i]$ for some $\alpha_i, \beta_i \in \mathbb{Z}$. Assume X and Y are not isomorphic as G-sets. This implies that there are some i's such that $\alpha_i \neq \beta_i$. Let $j = max(\{i \in \{1, 2, ..., n\} : \alpha_i \neq \beta_i\})$. Therefore $\sum_{i \geq j} \alpha_i |(G/K_i)^{K_j}| = |X^{K_j}| = |Y^{K_j}| = \sum_{i \geq j} \beta_i |(G/K_i)^{K_j}|$ so $0 = \sum_{i \geq j} (\alpha_i - \beta_i)|(G/K_i)^{K_j}| = (\alpha_j - \beta_j)|(G/K_j)^{K_j}| = (\alpha_j - \beta_j) r_j$. Since $r_j \neq 0$, $\alpha_j = \beta_j$ which is a contradiction. Hence $X \simeq Y$ and φ is injective.

Now we aim to show that if $f \in \varphi(\Omega(G))$, then for every $K \trianglelefteq H \leqslant G$ we have $\sum_{hK \in H/K} f(\langle h, K \rangle) \equiv 0 \mod [H : K]$. But $f \in \varphi(\Omega(G))$ implies that $f(\langle h, K \rangle) =$

$X^{\langle h,K\rangle} = (X^{\langle K\rangle})^{hK}$, so it sufficies to show that $\sum_{h\in H} X^{\langle h\rangle} \equiv 0 \mod |H|$ which is exactly what Lemma 2.5 says. Hence $\varphi(\Omega(G)) \subseteq \ker(\psi)$.

Let $u_i(K_j) := 1/r_i \cdot |(G/K_i)^{K_j}|$. We claim that $\{u_1, u_2, ..., u_n\}$ is a $\mathbb{Z}$-basis of $\Omega^*(G)$. By Lemma 2.6, $u_i(K_j) \in \mathbb{Z}$ for all j. Let $e_i \in \Omega^*(G)$ be such that

$$e_i(K_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

$\{e_1, e_2, ..., e_n\}$ gives a (standard) basis of $\Omega^*(G)$. Observe that

$$u_i = \sum_{j\leq i} 1/r_i \cdot |(G/K_i)^{K_j}| \cdot e_j = 1 \cdot e_i + \sum_{j<i} 1/r_i \cdot |(G/K_i)^{K_j}| \cdot e_j$$

i.e. there is an invertible upper-triangular matrix M whose all diagonal entries are 1 such that $[e_1\ e_2\ ...\ e_n] \cdot M = [u_1\ u_2\ ...\ u_n]$. Hence $\{u_1, u_2, ..., u_n\}$ is a $\mathbb{Z}$-basis of $\Omega^*(G)$. Moreover $r_1u_1, r_2u_2, ..., r_nu_n \in \varphi(\Omega(G))$. Hence $\Omega^*(G)/\varphi(\Omega(G)) \subseteq \bigoplus_{i=1}^{n} \mathbb{Z}/r_i\mathbb{Z}$.

$\psi(e_i) = \bar{e}_i = (0, 0, ..., \bar{1}, ..., 0)$ so since $\{e_1, e_2, ..., e_n\}$ is a basis of $\Omega^*(G)$ and $\{\bar{e}_1, \bar{e}_2, ..., \bar{e}_n\}$ is a basis of $\bigoplus_{i=1}^{n} \mathbb{Z}/r_i\mathbb{Z}$, we can conclude that ψ is surjective.

Finally, $|\Omega^*(G)/\varphi(\Omega(G))| \leq |\bigoplus_{i=1}^{n} \mathbb{Z}/r_i\mathbb{Z}| = |\psi(\Omega^*(G))| \leq |\Omega^*(G)/\ker(\psi)|$ and therefore $|\varphi(\Omega(G))| \geq |\ker(\psi)|$. Thus we have shown that φ is injective, ψ is surjective and $\varphi(\Omega(G)) = \ker(\psi)$. Hence the given sequence is exact. □

Corollary 2.9. *$\Omega^*(G)/\varphi(\Omega(G)) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/r_i\mathbb{Z}$.*

As we proved in Theorem 2.4, the Mark homomorphism sends the basis elements of $\Omega(G)$, namely $[G/H]$ assuming $H \in Cl(G)$, to $\sum_{K\leqslant H} |(G/H)^K| \cdot e_K$. However when we attempt to send the basis elements e_K's of $\Omega^*(G)$ to $\sum_{K\leqslant H} \lambda_H[G/H]$, the coefficients λ_H's usually are not integers. Because of that we will introduce a new concept to make desired calculations.

Definition 2.10. Let R be a commutative ring.

(i) The *Burnside algebra* of G over R is defined by $R\otimes_{\mathbb{Z}}\Omega(G)$ and denoted by $\Omega_R(G)$.
(ii) $\Omega^*_R(G) := \{f : Cl(G) \to R \mid f \text{ is a function}\}$.
(iii) The map $\varphi_R : \Omega_R(G) \to \Omega^*_R(G)$ determined by $\varphi_R(X) = f$ such that $f(K) = |X^K| \cdot 1_R$ is called the *extended mark homomorphism* of G over R.

Notice that φ_R is a homomorphism of R-algebras. Moreover by Theorem 2.4 it is injective. The following result indicates a condition for which φ_R is a bijection which means that we can make the desired calculations.

Theorem 2.11. *Let R be a commutative ring such that $|G|$ is a unit of R. Then the map $\varphi_R : \Omega_R(G) \to \Omega^*_R(G)$ is an isomorphism of R-algebras and*

$$\Omega_R(G) \cong \Omega^*_R(G) \cong R^{|Cl(G)|}.$$

Proof. Let $Cl(G)$ be the set of representatives of all conjugacy classes of G. Define

$$u_H(K) = \frac{1}{[N_G(H) : H]}|(G/H)^K|$$

for every $H, K \in Cl(G)$. Let $\{e_H : H \in Cl(G)\}$ be the natural basis of $\Omega_R^*(G)$. Then by repeating the argument in the proof of Theorem 2.4, $\{u_H : H \in Cl(G)\}$ is an R-basis of $\Omega_R^*(G)$. Since $|G|$ is a unit of R, $[N_G(H) : H]$ is also a unit for each $H \in Cl(G)$. Therefore by letting $b_H(K) = |(G/H)^K|$,

$$\{[N_G(H) : H] \cdot u_H : H \in Cl(G)\} = \{b_H : H \in Cl(G)\}$$

is an R-basis of $\Omega_R^*(G)$. Since $\{b_H : H \in Cl(G)\}$ is also an R-basis of $\Omega_R(G)$, we have the desired result. □

Corollary 2.12. $\Omega_{\mathbb{Q}}(G) \cong \Omega^*_{\mathbb{Q}}(G)$.

3. Gluck Idempotent Formula

Let $Cl(G)$ be the set of representatives of all conjugacy classes of a finite group G. Let

$$e_H(K) = \begin{cases} 1 & H \text{ is } G\text{-conjugate to } K \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_H(K) := 1/[N_G(H) : H] \cdot |(G/H)^K|.$$

Consider $\Omega_{\mathbb{Q}}(G) = \mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ the Burnside algebra of G over $\mathbb{Q}$. Notice that e_H's are the primitive idempotents of $\Omega_{\mathbb{Q}}(G)$ where each $H \in Cl(G)$. In light of Corollary 2.12 whenever we have $\varphi_{\mathbb{Q}}(\sum_{K \leqslant H} \lambda_H[G/H]) = e_K$, for some $\lambda_H \in \mathbb{Q}$. Aim of this chapter is to find out a formula for those λ_H's.

Example 3.1. In Example 2.8, we have found the marks of S_3. Therefore we can write

$$\begin{aligned} [S_3/1] &= 6e_1, \\ [S_3/C_2] &= 3e_1 + e_{C_2}, \\ [S_3/C_3] &= 2e_1 + 2e_{C_3}, \\ [S_3/S_3] &= e_1 + e_{C_2} + e_{C_3} + e_{S_3}. \end{aligned}$$

Using those we can calculate

$$\begin{aligned} e_1 &= 1/6[S_3/1], \\ e_{C_2} &= [S_3/C_2] - 1/2[S_3/1], \\ e_{C_3} &= 1/2[S_3/C_3] - 1/6[S_3/1], \\ e_{S_3} &= [S_3/S_3] - 1/2[S_3/C_3] - [S_3/C_2] + 1/2[S_3/1]. \end{aligned}$$

Instead of doing those calculations by hand, we will be using the idempotent formula we will find. To determine the desired formula for the primitive idempotents of $\Omega_{\mathbb{Q}}(G)$, known as Gluck Idempotent Formula, we will use some facts from combinatorial theory of partially ordered sets.

Definition 3.2. (i) A *partial order* is a binary relation on a set which is reflexive, antisymmetric and transitive.
(ii) A *partially ordered set* (or a *poset*) is a set P together with a specified partial order $\leq$ on P.
(iii) An *interval* of a poset P is defined by $[a,b] = \{x \in P : a \leq x \leq b\}$ where $a, b \in P$ with $a \leq b$.

The matrix

$$D_{i,j} = \begin{cases} 1 & i \leq j \\ 0 & \text{otherwise} \end{cases}$$

is called the *incidence matrix* of a poset P. Since its determinant is 1, it is invertible. So we can define $\mu(i,j) = (D^{-1})_{i,j}$ which is called the *Mobius function* of P. Consequently,

$$\sum_{i \leq k \leq j} \mu(i,k) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

The subgroup lattice of a group is a poset with the subgroup inclusion. Moreover the formula

$$(**) \qquad [G/H] = \sum_{K \in Cl(G)} |(G/H)^K| \cdot e_K$$

consists of a function defined on that poset. Fortunately, there is a classical result in partially ordered set theory as follows.

Theorem 3.3 (Mobius Inversion)**.** *Let $(P, \leq)$ be a poset, μ be its Mobius function and $f, g : P \to \mathbb{C}$ be such that $g(x) = \sum_{y \leq x} f(y)$. Then*

$$f(x) = \sum_{y \leq x} \mu(y,x) g(y).$$

Proof.

$$\begin{aligned} \sum_{y \leq x} \mu(y,x) g(y) = \sum_{y \leq x} \mu(y,x) (\sum_{z \leq y} f(z)) \\ = \sum_{z \leq y \leq x} \mu(y,x) f(z) = \sum_{z \leq x} f(z) (\sum_{z \leq y \leq x} \mu(y,x)) = f(x). \end{aligned}$$

□

Hence we are ready to state and prove the Gluck idempotent formula.

Theorem 3.4 (Gluck idempotent formula)**.** *Let $L(G)$ be the subgroup lattice of G, μ be the Mobius function of $(L(G), \leqslant)$ and*

$$e_H(K) = \begin{cases} 1 & H \text{ is } G\text{-conjugate to } K \\ 0 & \text{otherwise} \end{cases}$$

be a primitive idempotent of $\Omega_{\mathbb{Q}}(G)$ for each $H \in L(G)$. Then we have

$$e_H = \frac{1}{|N_G(H)|} \sum_{K \leqslant H} \mu(K,H) \cdot |K| \cdot [G/K].$$

To use Theorem 3.3, in the proof, we will manipulate (**) to get a function depended only K inside the sum.

Lemma 3.5. *Take the notation in the Theorem 3.4. We have*

$$[G/H] = \frac{1}{|H|} \sum_{K \leqslant H} |N_G(K)| \cdot e_K.$$

To prove this lemma we need the following result.

Lemma 3.6. *Let H and K be subgroups of a finite group G, then*

$$|(G/H)^K| = \frac{|N_G(K)|}{|H|} \cdot c(K,H)$$

where $c(K,H)$ is the number of G-conjugates of K contained in H.

Proof. Let $L(G)$ be the subgroup lattice of G. Define $\vartheta : L(G) \times L(G) \to \{0,1\}$ such that for every $A, B \in L(G)$,

$$\vartheta(A,B) = \begin{cases} 1 & A \leqslant B \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} |(G/H)^K| &= |\{gH \in G/H : K^g \subseteq H\}| = \frac{1}{|H|} \cdot |\{g \in G : K^g \subseteq H\}| \\ &= \frac{1}{|H|} \cdot \sum_{g \in G} \vartheta(K^g, H) = \frac{1}{|H|} \cdot \sum_{N_G(K)g \in G/N_G(K)} \sum_{ng \in N_G(K)g} \vartheta(K^{ng}, H). \end{aligned}$$

Since $n \in N_G(K)$ implies that $K^{ng} = K^g$ and the size of each coset of $N_G(K)$ is $|N_G(K)|$ we have

$$\sum_{ng \in N_G(K)g} \vartheta(K^{ng}, H) = \begin{cases} |N_G(K)| & K^g \leqslant H \\ 0 & \text{otherwise.} \end{cases}$$

Hence the result follows. □

Alternative Proof of Lemma 3.6. Let $\xi(A)$ be denote the number of G-conjugates of a subgroup A of G. H contains $c(K,H)$ conjugates of K. Since each conjugate of H contains equal number of conjugates of K, all conjugate of H contains $\xi(H) \cdot c(K,H)$ many conjugates of K if we treat duplicates of K as different. On the other hand each conjugates of K contained in exactly $n(K,H)$ many conjugates of H. Therefore for each conjugate of K, there are $n(K,H)$ duplicates in $\xi(H) \cdot c(K,H)$ many conjugates. Hence number of conjugates of K is

$$\xi(K) = \frac{\xi(H) \cdot c(K,H)}{n(K,H)}.$$

Notice that $\xi(A) = n(1,A) = \dfrac{|(G/A)^1|}{[N_G(A) : A]} = \dfrac{|G|}{|N_G(A)|}$ for every $A \leqslant G$, by Lemma 2.6. Therefore we have

$$\frac{N_G(H)}{N_G(K)} = \frac{c(K,H)}{n(K,H)}$$

Hence the result follows from Lemma 2.6. □

Proof of Lemma 3.5.

$$[G/H] = \sum_{K\in Cl(G)} |(G/H)^K| \cdot e_K = \sum_{K\in Cl(G), K\leqslant H} |(G/H)^K| \cdot e_K$$

$$= \sum_{K\leqslant H} \frac{1}{c(K,H)} |(G/H)^K| \cdot e_K = \frac{1}{|H|} \sum_{K\leqslant H} |N_G(K)| \cdot e_K.$$

□

We complete the proof of the Gluck idempotent formula.

Proof of Theorem 3.4. By Lemma 3.5 we have

$$|H| \cdot |G/H] = \sum_{K\leqslant H} |N_G(K)| \cdot e_K.$$

Hence apply Theorem 3.3 to get

$$|N_G(H)| \cdot e_H = \sum_{K\leqslant H} \mu(K,H) \cdot |K| \cdot [G/K].$$

as desired. □

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Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey

E-mail address: ahmet.kebeci@ug.bilkent.edu.tr