SCHUR-ZASSENHAUS THEOREM AND HALL SUBGROUPS

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Abstract. The goal of this project is to introduce Hall subgroups, to explain some tools for understanding Hall subgroups, including the Schur-Zassenhaus Theorem, and to introduce π-separable groups in light of these tools. First, a criterion for nilpotence is discussed. Then a detailed proof of Schur-Zassenhaus Theorem is given. After a short introduction to π-subgroups, Hall subgroups are introduced to lead to some results on π-separable and π-solvable groups. Similarities between the results on Hall subgroups and Sylow theorems are emphasized. Finally, relations between solvability and π-separability are presented.

Introduction

Let G be a finite group. A subgroup whose order is p^n, where n is the multiplicity of a prime p in the order of G, is called a *Sylow p-subgroup* of G. For a fixed prime dividing the order of the group, Sylow proved that such subgroups always exist and conjugate to each other([6, 1.6.16]). He also proved that the number of Sylow p-subgroups divides $\frac{|G|}{p^n}$ and it is congruent to 1 mod p. These results are known as the Sylow theorems.

Consider the dihedral group of order 30, $D_{30} = \langle x, a | a^{15} = x^2 = e, xax^{-1} = a^{-1}\rangle$. It contains 5 subgroups of order $6 = 2 \cdot 3$; each of them is isomorphic to D_6 and conjugate to each other. However the Sylow theorems do not give this result because D_6 is not a p-subgroup. So to examine such subgroups we need a new concept. To introduce this new concept we will need some preliminary definitions. Let π be a non-empty set of primes. A *π-number* is a positive integer whose prime divisors belong to π. (Note that 1 is a π-number for every π.) A group element whose order is a π-number is called a *π-element* and a group is called *π-group* if every element of it is a π-element. Notice that a finite group is a π-group if and only if its order is a π-number. In this project we will take it as a definition, since we will work on finite groups. A subgroup of G that is a π-group is called *π-subgroup*. One important thing about these definitions is that when we take $\pi = \{p\}$, where p is prime, we speak of p-elements and p-groups. In the above example, D_6 is a $\{2, 3\}$-subgroup of D_{30}. Moreover it is a maximal $\{2, 3\}$-subgroup. We will give a special name for such subgroups.

Definition 0.1. Let G be a group and let π be a set of primes. A maximal π-subgroup of G is called *Sylow π-subgroup* of G.

Now consider the alternating group, A_5, of order 60 and the set of primes $\pi = \{3, 5\}$. The Sylow $\{3, 5\}$-subgroups of A_5 are isomorphic to either C_3 or C_5, so not all of them are conjugate. Moreover the C_3 s have index 20 and the C_5 s have index 12. However, taking $\pi = \{p\}$, a Sylow p-subgroup is a π-group and its index is a π'-number where π' is the complement of π in the set of prime numbers. For π-subgroups we will have a similar definition. A π-subgroup will be called *Hall π-subgroup*, if its index is a π'-number. In Section 2, Hall subgroups are introduced in detail. A Hall $\{3, 5\}$-subgroup of A_5 should have index 4 and order 15, but A_5 does not have any such subgroup. So the existence of Hall subgroups is not guaranteed.

We investigate when a finite group has Hall π-subgroups and if they exist, when these subgroups are conjugate. For this cause, we state and prove the Schur-Zassenhaus theorem (Theorem 3.1) in Section 3. This theorem shows that if a normal Hall π-subgroup exists, then Hall π'-subgroups exist and are conjugate. Theorem 3.5 then shows that solvable groups have Hall subgroups and these subgroups are conjugate, without needing that normality condition.

In Section 4, we introduce π-separable groups. We see solvable groups as a first example of such groups. Then we collect some information about π-separable groups such as the upper π-series. Later we prove Theorem 4.11 and Theorem 4.13, which say that in π-separable groups, Hall subgroups always exist and are conjugate. This generalizes the result in solvable groups.

Finally, we prove Theorem 4.14, which shows that the existence of Hall p'-subgroups for every prime p dividing the order of the group implies solvability. We understand the connection between solvability and Hall subgroups. Moreover, this theorem provides a criterion for solvability in terms the existence of Hall subgroups as stated in Corollary 4.17.

Notations, definitions and results are mostly followed from Derek J.S. Robinson's book, [6].

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1. P. Hall's Criterion for Nilpotence

Let G be a finite group. If a normal subgroup of G and its quotient group are solvable then G is solvable. We cannot make this exact claim for nilpotence, but we have a similar statement. We look at a normal subgroup and the quotient group of its derived subgroup. If both of them are nilpotent we say G is nilpotent (Theorem 1.4). In this section, we will prove this statement. To complete the proof we will give a property of Frattini subgroup which is defined as follows.

Definition 1.1. Let G be a group. The *Frattini subgroup* of G is defined to be the intersection of all maximal subgroups of G and denoted by $\operatorname{Frat} G$ or $\Phi(G)$. For the case that G has no maximal subgroups, it is defined by $\operatorname{Frat} G = G$.

There is another way to see the Frattini subgroup. For this we define non-generators.

Definition 1.2. Let G be a group. An element g of G is called a *non-generator* or a *non-generating element* of G if whenever S is a generating set for G such that $g \in S$, $S \setminus \{g\}$ is also a generating set for G.

Note that $\operatorname{Frat} G$ is also the set of all non-generators of G (see [6, 5.2.12]). The required property of the Frattini subgroup for Theorem 1.4 is given as follows.

Lemma 1.3. *Let G be a finite group. If $N \trianglelefteq G$ then* $\operatorname{Frat} N \leqslant \operatorname{Frat} G$.

Proof. Since $\operatorname{Frat} N$ is a characteristic subgroup in N and $N \trianglelefteq G$, we have $\operatorname{Frat} N \trianglelefteq G$. Now assume that $\operatorname{Frat} N \nleqslant \operatorname{Frat} G$. Then, $\operatorname{Frat} N \nleqslant M$ for some maximal subgroup M of G. Then $G = M \cdot \operatorname{Frat} N$. Hence $N = N \cap G = N \cap (M \cdot \operatorname{Frat} N) = (N \cap M) \cdot \operatorname{Frat} N$. However $\operatorname{Frat} N$ is the set of non-generators of N, this implies that $N = N \cap M$. So $\operatorname{Frat} N \leqslant N \leqslant M$, which is a contradiction.

□

The reader can see [6, p. 135-137], for more detailed information about Frattini subgroups. Now we have all the tools to introduce a criterion for nilpotence as promised.

Theorem 1.4 (P. Hall's criterion for nilpotence)**.** *Let G be a finite group and $N \trianglelefteq G$. If N and G/N' are nilpotent, then G is nilpotent.*

Proof of Theorem 1.4. Let $N \trianglelefteq G$, N and G/N' be nilpotent and M be a maximal subgroup of N. Since N is nilpotent, M is normal and has prime index. Then N/M is a cyclic group, hence abelian. Therefore $N' \leqslant M$. But this is true for all maximal subgroups of N, so $N' \leqslant \operatorname{Frat} N$. Since $N \trianglelefteq G$, then by Lemma 1.3, $\operatorname{Frat} N \leqslant \operatorname{Frat} G$. So $N' \leqslant \operatorname{Frat} G$ and $(\operatorname{Frat} G)/N' \trianglelefteq G/N'$. Hence since G/N' is nilpotent, $G/\operatorname{Frat} G \cong (G/N')/((\operatorname{Frat} G)/N')$ is also nilpotent. Then every maximal subgroup of $G/\operatorname{Frat} G$ is normal (see [6, 5.2.4]). As $\operatorname{Frat} G$ is already contained in every maximal subgroup of G, every maximal subgroup of G is normal. Thus G is nilpotent (see [6, 5.2.4]).

□

Remark 1.5. Actually, Theorem 1.4 is true for all groups, not only for finite groups. Here we gave the proof for the finite case. For a proof including infinite groups, see [6, 5.2.10].

2. Hall Subgroups

One of the crucial subjects for this project is Hall subgroups. In this section, we give the definition of Hall subgroups and introduce some basic properties of normal π-subgroups. For more details about this topic, see [6, p. 252-253].

Definition 2.1. Let G be a finite group. A subgroup $H \leqslant G$ is called a *Hall π-subgroup* if H is a π-subgroup and $[G : H]$ is a π'-number.

Notice that a Hall $\{p\}$-subgroup is a Sylow p-subgroup so it always exists. However, when π contains more than one prime the situation changes. As we see in Example 2.2, the existence of such subgroups is not guaranteed. On the other hand Sylow π-subgroups (see Definition 0.1) always exist, by definition. It is easy to see that if H is a Hall π-subgroup of a finite group G, then H is also a Sylow π-subgroup. On the other hand let S be a Sylow π-subgroup of a finite group G_0. If Hall π-subgroups exist in G_0, then S is one of them. So actually for a finite group, Hall π-subgroups exists if and only if Hall and Sylow π-subgroups are same.

Example 2.2. Consider the alternating group, A_5, of order 60 and the dihedral group, D_{210}, of order 210.

(i) The Hall $\{2,3\}$-subgroups of A_5 are isomorphic to A_4.
(ii) There is no Hall $\{2,5\}$-subgroup of A_5. The Sylow $\{2,5\}$-subgroups of A_5 are isomorphic to D_{10}, the dihedral group of order 10.
(iii) The Hall $\{2,3,7\}$-subgroups of D_{210} are isomorphic to D_{42}.

Let H, K be π-subgroups of a group G and $K \triangleleft G$. One can say that $H \cap K$ and HK/K are π-subgroups. Therefore HK is a π-subgroup. Similarly, if we take two normal π-subgroups then their product is also a normal π-subgroup. Hence there is the unique maximum normal π-subgroup of G. We will denote this subgroup by $O_\pi(G)$. Note that $O_\pi(G)$ is a characteristic subgroup of G.

By definition, any normal π-subgroup of G is contained in $O_\pi(G)$. The next theorem shows that any subnormal π-subgroup of G is also contained in $O_\pi(G)$.

Theorem 2.3. *If H is a subnormal π-subgroup of a finite group G, then $H \leqslant O_\pi(G)$.*

Proof. If H is subnormal, then there is a subnormal series from H to G,

$$H = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft H_{n-1} \triangleleft H_n = G.$$

$H \leqslant O_\pi(H_1)$, since H is a normal π-subgroup of H_1. Assume $H \leqslant O_\pi(H_i)$ for some $i < n$. Since $O_\pi(H_i)$ is characteristic in H_i, $O_\pi(H_i)$ is normal in H_{i+1}. Then, $O_\pi(H_i) \leqslant O_\pi(H_{i+1})$ and by assumption, $H \leqslant O_\pi(H_{i+1})$. Therefore by induction, $H \leqslant O_\pi(H_i)$ for every i. Thus, $H \leqslant O_\pi(H_n) = O_\pi(G)$.

□

Theorem 2.4. *Let G be a finite group. $O_\pi(G)$ is the intersection of all the Sylow π-subgroup of G.*

Proof. Let $R := O_\pi(G)$ and let S be a Sylow π-subgroup of G. Then RS is a π-group and therefore $R \leqslant S$. On the other hand, the intersection of all the Sylow π-subgroup of G is a normal π-subgroup of G, so it is contained in $O_\pi(G)$.

□

3. The Schur-Zassenhaus Theorem

In this section, we will prove the Schur-Zassenhaus Theorem and some of its corollaries. Since it plays a critical role in the understanding Hall subgroups, the Schur-Zassenhaus Theorem is one of the major parts of this project. For a more detailed explanation about its proof, see [1, Section 6 and 9].

Theorem 3.1 (Schur-Zassenhaus theorem)**.** *Let G be a finite group and let $N \trianglelefteq G$. Assume that $|N| = n$ and $[G : N] = m$ are relatively prime. Then, G contains a subgroup of order m and any two such subgroups are conjugate in G.*

To prove the Schur-Zassenhaus Theorem, some results from the theory of group extensions are required. The following lemma provides us all the information we need.

Definition 3.2. Let G be a group, N be a normal subgroup of G and Q be isomorphic to the quotient group G/N. A *group extension* of Q by N is a short exact sequence

$$1 \to N \xrightarrow{\epsilon} G \xrightarrow{\nu} Q \to 1$$

i.e., $\mathrm{Im}(\epsilon) = \ker(\nu)$, $\mathrm{Im}(\nu) = Q$ and $\ker(\epsilon) = 1$.

Lemma 3.3. [1, Proposition 6.3] *Let* $1 \to N \xrightarrow{\epsilon} G \xrightarrow{\nu} Q \to 1$ *be a group extension of* Q *by* N. *For each* $x \in Q$, *let* $t_x \in G$ *be such that* $\nu(t_x) = x$. *Then,*

(a) For every $g \in G$, *there exist unique elements* $x \in Q$ *and* $a \in N$ *such that* $g = t_x\epsilon(a)$.
(b) For every $x \in Q$ *and* $a \in N$, *there exists a unique element* $\alpha_x(a) \in N$ *such that* $\epsilon(\alpha_x(a)) = t_x\epsilon(a)t_x^{-1}$.
(c) For every $x, y \in Q$, *there exists a unique element* $k(x, y) \in N$ *such that* $t_xt_y = \epsilon(k(x, y))t_{xy}$. *Moreover,* $\alpha_x \circ \alpha_y = k(x, y)\alpha_{xy}k(x, y)^{-1}$.
(d) For every $x, y, z \in Q$, $k(x, y)k(xy, z) = \alpha_x(k(y, z))k(x, yz)$.
(e) Let also $t'_x \in G$ *be such that* $\nu(t'_x) = x$, *for each* $x \in Q$. *Then there exists a unique function* $g : Q \to N$ *such that* $t'_x = t_x \cdot \epsilon(g(x))$, *for each* $x \in Q$.
Also for every $x, y \in Q$, *if* α' *and* k' *are constructed from* t'_x, *then* $\alpha'_x = f(x)\alpha_xf(x)^{-1}$ *and* $k'(x, y) = f(x)\alpha_x(f(y))k(x, y)f(xy)^{-1}$ *where* $f : Q \to N$ *is defined by* $f(x) := \alpha_x(g(x))$.

Proof of Theorem 3.1. Case 1: Assume N is an abelian group.

(Existence) Let $Q := G/N$ and consider the extension of Q by N;

$$1 \to N \xrightarrow{\epsilon} G \xrightarrow{\nu} Q \to 1$$

For all $x \in Q$ let $t_x \in G$ be such that $\nu(t_x) = x$. Define $a \in N$, $\alpha_x(a) \in N$ and $k(x, y) \in N$ as in Lemma 3.3. Let $c(x, y) := \epsilon(k(x, y)) \in \epsilon(N)$. Then $t_x \cdot t_y = c(x, y) \cdot t_{xy}$. By Lemma 3.3(d), $k(x, y)k(xy, z) = \alpha_x(k(y, z)k(x, yz))$. Then by applying ϵ to each side, we get

$$c(x, y)c(xy, z) = {}^{t_x}c(y, z)c(x, yz),$$

since we have $\epsilon(\alpha_x(a)) = {}^{t_x}\epsilon(a)$, for all $a \in N$ from Lemma 3.3(c). Let $g(\zeta) := \prod_{z \in Q} c(z, \zeta)$. Because $c(z, \zeta) \in \epsilon(N)$, we have $g(\zeta) \in \epsilon(N)$ for every $\zeta \in Q$. Then since $\epsilon(N)$ is abelian, from the last equation,

$$\prod_{z \in Q} c(x, y) \prod_{z \in Q} c(xy, z) = \prod_{z \in Q} {}^{t_x}c(y, z) \prod_{z \in Q} c(x, yz).$$

Hence we have $[c(x, y)]^m g(xy) = {}^{t_x}g(y)g(x)$.

Since $gcd(n, m) = 1$, there exists $r \in \mathbb{Z}$ such that $rm \equiv 1\ (mod\ n)$. Set $h(\zeta) := g(\zeta)^r$, $y_\alpha := t_\alpha^{-1}h(\alpha)$ and $K := \{y_\alpha : \alpha \in Q\}$. K is a subset of G. We will prove that K is a subgroup of order m and K will be the subgroup we are looking for. Let $y_\alpha, y_\beta \in K$. Then,

$$y_\beta y_\alpha = t_\beta^{-1}h(\beta)t_\alpha^{-1}h(\alpha) = t_\beta^{-1}t_\alpha^{-1} \cdot^{t_\alpha} h(\beta)h(\alpha) = t_{\alpha\beta}^{-1} \cdot c(\alpha, \beta)^{-1} \cdot^{t_\alpha} h(\beta)h(\alpha).$$

But,

$$\begin{aligned} {}^{t_\alpha}h(\beta)h(\alpha) &= {}^{t_\alpha}(g(\beta)^r)g(\alpha)^r = ({}^{t_\alpha}g(\beta)g(\alpha))^r \\ &= ([c(\alpha, \beta)]^m g(\alpha\beta))^r = [c(\alpha, \beta)]^{rm} \cdot g(\alpha\beta)^r = c(\alpha, \beta) \cdot h(\alpha\beta). \end{aligned}$$

So,

$$y_\beta y_\alpha = t_{\alpha\beta}^{-1} \cdot c(\alpha, \beta)^{-1} \cdot c(\alpha, \beta) \cdot h(\alpha\beta) = t_{\alpha\beta}^{-1} \cdot h(\alpha\beta) = y_{\alpha\beta}.$$

So K is a subgroup of G. Now assume $y_\alpha = y_\beta$. Then $y_\alpha y_\beta^{-1} = 1$. However, our last result implies that $y_{\beta^{-1}\alpha} = 1$. By letting $\gamma := \beta^{-1}\alpha$, we get $y_\gamma = 1$ and then $t_\gamma^{-1}h(\gamma) = 1$. But $h(\gamma) \in \epsilon(N)$, so $t_\gamma \in \epsilon(N)$. Therefore $\gamma = \nu(t_\gamma) = 1$. Thus, $\alpha = \beta$. So for every α in Q, we get different y_α, then $|K| = |Q| = m$. K's being a subgroup of order m in G completes the existence part of this case.

(Conjugacy) Let H and H^* be subgroups of G with $|H| = |H^*| = m$. Then $G = HN = H^*N$ and $H \cap N = 1 = H^* \cap N$. Let $Q := G/N$ again. So $Q = HN/N = H^*N/N$. Take the same extension again:

$$1 \to N \xrightarrow{\epsilon} G \xrightarrow{\nu} Q \to 1$$

For all $x \in Q$, let $u_x \in H$ and $u_x^* \in H^*$ be such that $\nu(u_x) = x$ and $\nu(u_x^*) = x$. By Lemma 3.3(e), $u_x^* = u_x \cdot \epsilon(g(x))$ for some unique map $g : Q \to N$. Let $a(x) := \epsilon(g(x))$. So $a : Q \to \epsilon(N)$. Then, $u_x^* = u_x \cdot a(x)$. Also,

$$\begin{aligned} u_{xy}^* = c(x,y)^{-1}u_x^*u_y^* &= c(x,y)^{-1}u_x a(x) u_y a(y) \\ &= c(x,y)^{-1}u_x u_y \cdot^{u_y^{-1}} a(x)a(y) = u_{xy} \cdot^{u_y^{-1}} a(x)a(y) \end{aligned}$$

So, $a(xy) = {}^{u_y^{-1}}a(x)a(y)$. Let $b := \prod_{x\in Q} a(x)$. Then by taking products in the above equation and using that $\epsilon(N)$ is abelian,

$$b =^{u_y^{-1}} b \cdot a(y)^m.$$

Since $gcd(n,m) = 1$, there exists $r \in \mathbb{Z}$ such that $rm \equiv 1 \ (mod\ n)$. So let $c := b^r$. Then $c^m = b^{rm} = b$, so by the above equation $c^m = {}^{u_y^{-1}}(c^m) \cdot a(y)^m$ and $c = {}^{u_y^{-1}}c \cdot a(y)$ therefore $a(y) = {}^{u_y^{-1}}(c^{-1}) \cdot c$. Thus $u_y^* = u_y a(y) = u_y \cdot^{u_y^{-1}} (c^{-1}) \cdot c = c^{-1}u_y c$. Hence $H^* = c^{-1}Hc$.

General Case: Assume N is any normal subgroup.

(Existence) Assume that existence part is not correct and G is a counterexample of minimal order. Let $P \in Syl_p(N)$. By the Frattini Argument (see [6, 5.2.14]), $N_G(P)N = G$. Assume $N_G(P) < G$. By Second Isomorphism Theorem, $N \cap N_G(P) \lhd N_G(P)$ with $[N_G(P) : N \cap N_G(P)] = [N_G(P)N : N] = [G : N] = m$, relatively prime to $|N \cap N_G(P)|$. Then by minimality of G, there exists $H \leqslant N_G(P)$ such that $|H| = m$. But this implies that $H \leqslant G$ which is a contradiction by assumption. Therefore $N_G(P) = P$ and $P \trianglelefteq G$, so $P \trianglelefteq N$. But this is for every Sylow subgroup of N. Thus, N is nilpotent so $Z(N)$ is non-trivial. Let $\overline{G} := G/Z(N)$. Also let $\overline{N}$ be the image of N in $\overline{G}$. So by minimality of G, there exists $\overline{H} \leqslant \overline{G}$ such that $|\overline{H}| = [\overline{G} : \overline{N}] = [G : N] = m$. Let H be the inverse image of $\overline{H}$. Assume $H < G$. By minimality of G, there exists $K \leqslant H$ such that $|K| = [H : Z(N)] = |\overline{H}| = m$. However K is also a subgroup of G with order m. Hence this contradicts G being a counterexample. Therefore $G = H$. Hence $\overline{G} = \overline{H}$ and $\overline{N} = 1$. Thus $N = Z(N)$ which leads that N is abelian. Finally by the abelian case, we have existence.

(Conjugacy) Let $H, H^* \leqslant G$ with $|H| = |H^*| = m$. Assume that H is not conjugate to H^* and G is a counterexample of minimal order.

Case (i): Suppose that N is solvable.

Note that $N' \neq N$, by solvability of N, and $N' \neq 1$, by abelian case. We will consider G/N'. Since N/N' is an abelian normal subgroup of G/N', by the abelian case, HN'/N' and H^*N'/N' are conjugate in G/N'. Then ${}^gH \leqslant H^*N'$, for some $g \in G$. Since N' is a proper subgroup of N, we know that H^*N' is a proper subgroup of $H^*N = G$. It follows by minimality that ${}^gH$ and H^* are conjugate in H^*N' and therefore in G. Thus, H and H^* are conjugate in G.

Case (ii): Suppose that G/N is solvable.

Set $\pi := \{p \text{ prime} : p \mid m\}$ and $R := O_\pi(G)$. Since H and H^* are Sylow π-subgroups, $R \leqslant H \cap H^*$ by Theorem 2.4. Now consider G/R. Since $O_\pi(G/R) = 1$, we may suppose that $R = 1$. Let L/N be a minimal normal subgroup of G/N. Then since G/N is solvable, L/N is an elementary abelian p-group (see [5, Theorem 6.13]) for some $p \in \pi$. Also, $[L : H \cap L] = [LH : H] = [G : H] = n$ which is a p'-number. Therefore, $H \cap L \in Syl_p(L)$. Similarly, $H^* \cap L \in Syl_p(L)$. By the Sylow theorems, there exists $g \in G$ such that $H \cap L = {}^g(H^* \cap L) = {}^gH^* \cap {}^gL = {}^gH^* \cap L \lhd {}^gH^*$. So, $H \cap L \lhd \langle H, {}^gH^* \rangle$. Now let $J := \langle H, {}^gH^* \rangle$. Assume $J = G$. Then $H \cap L \lhd G$ hence $H \cap L \leqslant R = 1$ so $L/N = 1$. But this contradicts L/N's being a minimal normal subgroup of G/N. Then $J < G$. Finally use minimality to conclude that H and ${}^gH^*$ are conjugate in J and thus in G. Hence H and H^* are conjugate in G.

Since $|N|$ and $|G/N|$ are relatively prime, at least one of them must be odd. So Feit-Thompson Theorem [3] guarantees that at least one of N and G/N is solvable. This completes the proof.

□

Corollary 3.4. *Let G be a finite group and let $N \trianglelefteq G$. Assume that $|N| = n$ and $[G : N] = m$ are relatively prime. Let m_1 be a divisor of m. Then a subgroup of G with order m_1 is contained in a subgroup of order m, whose existence is guaranteed by the Schur-Zassenhaus theorem.*

Proof. Let $|M| = m$, $|M_1| = m_1$ and $M, M_1 < G$. Then by the Schur-Zassenhaus Theorem, $G = MN$. Then $M_1N = M_1N \cap G = M_1N \cap MN = (M_1N \cap M)N$. Since $(M_1N \cap M) \cap N \leqslant M \cap N = 1$, we have $|M_1N \cap M| = |(M_1N \cap M)N : N| = |M_1N : N| = |M_1| = m_1$. Then by applying the Schur-Zassenhaus Theorem to M_1N, there exists $g \in G$ such that $M_1 = {}^g(M_1N \cap M) \leqslant {}^gM$, whose order is m.

□

Theorem 3.1 shows that if there is a normal Hall π-subgroup in a finite group then Hall π'-subgroups exist and moreover, they are conjugate. By using Theorem 3.1, we will prove another result. This time, we remove normal subgroup from conditions but add solvability condition. This theorem is taken from [7, Theorem 2].

Theorem 3.5 (Extended Sylow theorem for solvable groups)**.** *Let G be a finite solvable group and let $|G| = mn$ such that m and n are relatively prime.Then, G contains a subgroup of order m and any two of such groups are conjugate in G.*

Proof. Let K be a minimal normal subgroup of G. Then K is an elementary abelian p-group (see [5, Theorem 6.13]) for some p dividing $|G|$. Then either $p|m$ or $p|n$.

(Existence) Assume that existence part is not true and let G be the minimal counterexample by order.

Case 1: If $p|m$, then since K is nontrivial, $|G/K| < |G|$. Also $|G/K| = n \cdot \frac{m}{|K|}$ where n and $\frac{m}{|K|}$ are relatively prime. So by minimality, G/K has a subgroup D/K of order $m/|K|$. Then $D \leqslant G$ with $|D| = m$.

Case 2: If $p|n$, then again since K is nontrivial, $|G/K| < |G|$ and $|G/K| = m \cdot \frac{n}{|K|}$ where m and $\frac{n}{|K|}$ are relatively prime. Similarly by minimality, G/K has a subgroup D/K of order m. Then by Schur-Zassenhaus Theorem, D has a subgroup D_0 such that $|D_0| = m$. Then $D_0 \leqslant G$.

(Conjugacy) Again, assume that conjugacy part is not true and let G be the minimal counterexample by order. Let $D_1, D_2 \leqslant G$ such that $|D_1| = |D_2| = m$.

Case 1: If $p|m$, then since $p \nmid n = \frac{|G|}{|D_1|}$, we have $p \nmid \frac{|D_1K|}{|D_1|} = \frac{|K|}{|D_1 \cap K|} = \frac{p^a}{|D_1 \cap K|}$ for some $a \in \mathbb{N}$. Then $|D_1 \cap K| = p^a = |K|$. So $D_1 \cap K = K$. Thus $K \leqslant D_1$. Likewise $K \leqslant D_2$.

Then by minimality, D_1/K and D_2/K are conjugate in G/K. Thus D_1 and D_2 are conjugate in G.

Case 2: If $p|n$, then $D_1 \cap K = 1$. Then $|D_1K/K| = |D_1| = m$. Likewise $|D_2K/K| = m$. So by minimality, D_1K/K and D_2K/K are conjugate in G/K. Then there is some $g \in G$ such that ${}^gD_1 \leqslant D_2K$. Then by Schur-Zassenhaus Theorem, ${}^gD_1$ and D_2 are conjugate in D_2K and hence in G. Thus, D_1 and D_2 are conjugate in G.

□

4. π-Separable Groups

In this section, we will introduce π-separable groups and π-solvable groups. We observe that there is a relationship between π-separability and Hall π-subgroups. We find some satisfying answers to our question, that is, which groups do have Hall subgroups. We will see that π-Separable groups is an answer to this question. The reader can find detailed information about this section in [6, Section 9.1].

For this section, every group will be assumed finite and π will be a set of primes, unless stated otherwise.

Definition 4.1. A π-*series* of G is a subnormal series such that each factor is either a π-group or a π'-group.

In addition, a π-series is called *p-series* when $\pi = \{p\}$. So the factor of p-series are either p-groups or have orders relatively prime to p.

Definition 4.2. A group G is called π-*separable* if there exists a π-series of G.

Proposition 4.3. *A finite solvable group is* π*-separable for all* π*.*

Proof. Let G be a finite solvable group and let π be a set of primes. Take the derived series of G.

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq ... \trianglerighteq G^{(n-1)} \trianglerighteq G^{(n)} = 1$$

Let $0 \leqslant i \leqslant n-1$. Since $G^{(i)}/G^{(i+1)}$ is abelian, it has a Hall π-subgroup, say $H_i/G^{(i+1)}$. Also $H_i/G^{(i+1)} \trianglelefteq G^{(i)}/G^{(i+1)}$. Then $G^{(i+1)} \trianglelefteq H_i \trianglelefteq G^{(i)}$ and $G^{(i)}/H_i$ is a

π'-group. Thus,

$$G = G^{(0)} \trianglerighteq H_0 \trianglerighteq G^{(1)} \trianglerighteq H_1 \trianglerighteq ... \trianglerighteq G^{(n-1)} \trianglerighteq H_{n-1} \trianglerighteq G^{(n)} = 1$$

is a π-series of G.

□

Definition 4.4. Let G be a finite group and let π be a set of primes.

(i) G is called *p-solvable* if G has a p-series.
(ii) G is called *π-solvable* if G has a π-series such that each factor is either a π'-group or a p-group for some $p \in \pi$.

Note that (ii) is a generalization of (i). A p-solvable group is π-solvable and π-separable for $\pi = \{p\}$.

Also, note that every subgroup and every image of a π-separable [π-solvable] group is π-separable [π-solvable].

Proposition 4.5. *A finite group is solvable if and only if it is p-solvable for every p prime dividing the order of the group.*

Proof. According to Proposition 4.3 by taking $\pi = \{p\}$, a solvable finite group is p-solvable for every p.

Now let G be a p-solvable group for every prime p dividing its order. Let $\pi(G)$ be the set of prime divisors of $|G|$. We will induct on the cardinality of $\pi(G)$. If $\pi(G)$ has only one element then G is a p-group, so it is solvable. Assume that a group H is solvable when $\pi(H)$ has fewer than n elements. Let $\pi(G)$ has n elements, namely, $p_1, p_2, \ldots, p_n$. Since G is p_1-solvable, it has a π-series such that each factor is either a p_1-group or $p_1{}'$-group. Factors that are p_1-groups are solvable. Other factors, being $p_1{}'$-groups, are $\{p_2, p_3, \ldots, p_n\}$-groups. So the number of prime divisors of their orders are less than n. Also they are p_i-solvable for $2 \leq i \leq n$, since G is p_i-solvable. Then by the inductive step, they are solvable. Thus each factor of this π-series is solvable, hence G is solvable.

□

To understand π-separable groups better, we will have a notion of the upper π-series. This concept will provide us a characterization for π-separable groups like derived series of solvable groups or central series of nilpotent groups.

Let G be a finite group. It is obvious that $O_\pi(\overline{G}) = 1$ where $\overline{G} = G/O_\pi(G)$. We consider $O_{\pi'}(\overline{G})$ and denote its preimage in G by $O_{\pi,\pi'}(G)$. Similarly we define $O_{\pi,\pi',\pi}(G)$ to be the inverse image of $O_\pi(G/O_{\pi,\pi'}(G))$ in G. Continuing this way we obtain a series of characteristic subgroups

$$1 \trianglelefteq O_\pi(G) \trianglelefteq O_{\pi,\pi'}(G) \trianglelefteq O_{\pi,\pi',\pi}(G) \trianglelefteq O_{\pi,\pi',\pi,\pi'}(G) \trianglelefteq \ldots$$

where each factor is either a π-group or a π'-group. As we define formally below, this series is going to be called the upper π-series of G.

Definition 4.6. Let G be a group. *The upper π-series* of G is defined to be the π-series

$$1 = P_0 \triangleleft N_0 \triangleleft P_1 \triangleleft N_1 \triangleleft \ldots \triangleleft P_m \triangleleft N_m \triangleleft \ldots$$

such that $N_i/P_i = O_\pi(G/P_i)$ and $P_{i+1}/N_i = O_{\pi'}(G/N_i)$.

Lemma 4.7. *Let G be a π-separable group and let $1 = H_0 \lhd K_0 \lhd H_1 \lhd K_1 \lhd \ldots \lhd H_m \lhd K_m = G$ be a π-series of G such that K_i/H_i is a π-group and H_{i+1}/K_i is a π'-group for each i. Then $H_i \leqslant P_i$ and $K_i \leqslant N_i$ where $\{P_i\}$ and $\{N_i\}$ are the terms of the upper π-series of G as above. In particular, $N_m = G$.*

Proof. First we will prove that $H_i \leqslant P_i$ for every i, by induction. For the base case, $1 = H_0 \leqslant P_0 = 1$. Assume $H_i \leqslant P_i$ for a fixed i. We know that K_i is subnormal in G and P_i is normal in G, so K_iP_i is subnormal in G. $K_iP_i/P_i \cong K_i/(K_i \cap P_i)$ is a π'-group. So K_iP_i/P_i is a subnormal π'-subgroup of G/P_i. Hence by Theorem 2.3, $K_iP_i/P_i \leqslant O_{\pi'}(G/P_i) = N_i/P_i$. Thus, $K_i \leqslant N_i$. By the same arguments, $H_{i+1}N_i/N_i$ is a subnormal π'-subgroup of G/N_i. So again by Theorem 2.3, $H_{i+1}N_i/N_i \leqslant O_{\pi'}(G/N_i) = P_{i+1}/N_i$. Hence, $H_{i+1} \leqslant P_{i+1}$ and we are done.

We have already proved that $H_i \leqslant P_i$ implies $K_i \leqslant N_i$.

□

Corollary 4.8. *A finite group G is π-separable if and only if the upper π-series of G terminates in G.*

Definition 4.9. Let G be a group.

(i) A *chief series* of G is a normal series $G = N_n \trianglerighteq N_{n-1} \trianglerighteq \ldots \trianglerighteq N_1 \trianglerighteq N_0 = 1$ such that, for each i, N_i/N_{i-1} is a minimal normal subgroup of G/N_{i-1}.
(ii) A *composition series* of G is a subnormal series such that each factor is simple.

Having these definitions, next theorem states that there are some equivalent ways to define π-separable groups.

Theorem 4.10. *Let G be a finite group. The following conditions are equivalent.*

(i) G is π-separable.
(ii) G is π'-separable.
(iii) There is a π-series for G that is a normal series.
(iv) There is a π-series for G that is a characteristic series.
(v) The upper π-series of G terminates in G.
(vi) The upper π'-series of G terminates in G.
(vii) Any chief series of G is a π-series.
(viii) Any composition series of G is a π-series.

Proof. By definition, (i) and (ii) are equivalent. By 4.8, (i) and (v) are equivalent. Likewise (ii) and (vi) are equivalent. Hence (i), (ii), (v) and (vi) are equivalent. Since the upper π-series is a characteristic series, (v) implies (iv). Also any characteristic series is a normal series, so (iv) implies (iii). Also (iii) implies (i) by definition. Thus, (i), (ii), (iii), (iv), (v) and (vi) are equivalent.

Assume G is π-separable. Let

$$G = N_n \trianglerighteq N_{n-1} \trianglerighteq \ldots \trianglerighteq N_1 \trianglerighteq N_0 = 1$$

be a chief series of G. So N_i is a normal subgroup of G and N_i/N_{i-1} is characteristically simple for all i. Also since G is π-separable, N_i/N_{i-1} is π-separable too. So $O_\pi(N_i/N_{i-1}) = 1$ and $O_{\pi'}(N_i/N_{i-1}) = N_i/N_{i-1}$ or vice versa. Then N_i/N_{i-1} is either a π-group or π'-group. Thus, (i) implies (vii). Also, (vii) implies (i), by definition. So (i) and (vii) are equivalent.

Again assume G is π-separable. Then every composition factor of G is π-separable, but they are also simple. So each of them is either a π-group or π'-group. Then (i) implies $(viii)$. Again the other way around comes from the definition. Thus (i) and $(viii)$ are equivalent.

□

Next results (Theorem 4.11 and Theorem 4.13) can be found in [6, 9.1.6]. These results, being similar with the Sylow theorems, are some of the most important properties of π-separable groups.

Theorem 4.11. *Let G be a π-separable group and let P be a Sylow π-subgroup of G. Then P is a Hall π-subgroup of G.*

Proof. We will do induction on the order of G. If its order is 1, we are done. Assume that the theorem is true for every group whose order is smaller than G.

Let $R := O_\pi(G)$. We will have two cases.

Case 1: R is non-trivial.

Then G/R is a proper quotient group of G. Since R is characteristic in G, we have $R \trianglelefteq P$. Then P/R is a Sylow π-subgroup of G/R. Also G/R is π-separable, since G is π-separable. Then by assumption on the inductive step, P/R is a Hall π-subgroup of G/R. Then $[G/R : P/R] = [G : P]$ is a π'-number. So P is a Hall π-subgroup of G.

Case 2: R is trivial.

Let $S := O_{\pi'}(G)$. If S is trivial, then the upper π-series of G only contains trivial group, which contradicts G's being π-separable and non-trivial. So S is non-trivial. Then G/S is a proper subgroup of G. Also by the second isomorphism theorem, $PS/S \cong P/(P \cap S) = P$ since $P \cap S = 1$. So PS/S is a π-subgroup of G/S. Then PS/S is contained in a Sylow π-subgroup of G/S, say Q/S, which is a Hall π-subgroup of G/S by the assumption on the inductive step. Recall that P is a subgroup of Q. By Corollary 3.4, $P \leqslant P^*$ where P^* is a Hall π-subgroup of Q. But P is a Sylow π-subgroup of Q, thus $P = P^*$. So $[G : Q]$ is a π'-number and P is a Hall π-subgroup of Q. Since $[G : Q]$ is a π'-number, P is a Hall π-subgroup of G.

□

Corollary 4.12. *Every π-subgroup of a finite π-separable group is contained in a Hall π-subgroup of that group.*

Theorem 4.13. *Let the finite group G be π-separable. Then any two Hall π-subgroups of G are conjugate in G.*

Proof. Let P and Q be two Hall π-subgroups of G. We will use induction on the order of G. If its order is 1, we are done. Assume that the theorem is true for every group whose order is smaller than G.

Let $R := O_\pi(G)$ and $S := O_{\pi'}(G)$. If R is non-trivial, then by induction P/R and Q/R are conjugate in G/R and we are done. So we may assume that $R = 1$. Then S is non-trivial since G is π-separable. Then by induction PS/S and QS/S are conjugate in G/S. Then for some $g \in G$, we have ${}^gQ \leqslant PS$. Then by Theorem 3.1, P and ${}^gQ$ are conjugate in PS and hence P and Q are conjugate in G. □

Corollary 4.12 and Theorem 4.13 are similar to the Sylow theorems. However in this case, we have these theorems only for finite π-separable groups while the Sylow theorems hold for all finite groups. Therefore one can say that π-groups are a generalization of p-groups and Hall subgroups are an extension of Sylow p-subgroups.

Note that Theorem 3.5 is actually an immediate consequence of Corollary 4.12 and Theorem 4.13 because every finite solvable group is π-separable as we proved in Proposition 4.3. Remark that in the language of Hall subgroups, Theorem 3.5 says that for every π, every π-subgroup of a finite solvable group G is contained in a Hall π-subgroup of G and all Hall π-subgroups, for any fixed π, of G are conjugate to each other. Now we investigate if the converse of this statement holds. The following theorem shows that it holds.

Theorem 4.14. *Let G be a finite group and suppose that for every prime p there exists a Hall p'-subgroup of G. Then G is solvable.*

To prove this theorem, we will need Burnside's $p^\alpha q^\beta$ theorem. We will only state this theorem without giving a proof because it requires some tools we did not mention in this project.

Theorem 4.15 (Burnside's $p^\alpha q^\beta$ theorem, [2])**.** *Every group whose order is of the form $p^\alpha q^\beta$ is solvable.*

Proof of Theorem 4.14 also requires some index calculations. To help us to do this calculations we have the following lemma.

Lemma 4.16. *Let H and K be subgroups of a finite group G whose indices are relatively prime. Then $[G : H \cap K] = [G : H] \cdot [G : K]$.*

Proof. Because $([G : H], [G : K]) = 1$, we have $G = HK$. Therefore,

$$\frac{|G|}{|H \cap K|} = \frac{|G| \cdot |HK|}{|H| \cdot |K|} = [G : H] \cdot [G : K] \cdot \frac{|HK|}{|G|} = [G : H] \cdot [G : K].$$

□

Proof of Theorem 4.14. Assume that the theorem is not true and that G is the minimal counterexample by order. Let $|G| = p_1^{l_1} \cdot p_2^{l_2} \dots p_k^{l_k}$ where p_i's are distinct primes and l_i's are positive integers. If $k = 1$, then G is a p_1-group so it is solvable. If $k = 2$, then by Theorem 4.15, G is solvable. So $k > 2$. Let G_i be a Hall p_i'-subgroup of G. Then, $[G : G_i] = p_i^{l_i}$. Let $H := G_3 \cap G_4 \cap \dots \cap G_k$. By Lemma 4.16, $[G : H] = \prod_{i=3}^{k} p_i^{l_i}$. So $|H| = p_1^{l_1} p_2^{l_2}$. By Theorem 4.15, H is solvable. Let M be a minimal normal subgroup of H. Then M is an elementary abelian q-group where $q \in \{p_1, p_2\}$ (see [5, Theorem 6.13]). Without loss of generality let $q = p_1$. Similarly, by Lemma 4.16, $[G : H \cap G_2] = \prod_{i=2}^{k} p_i^{l_i}$, so $|H \cap G_2| = p_1^{l_1}$. So $H \cap G_2$ is a Sylow p_1-subgroup of H. Hence $M \leqslant (H \cap G_2) \leqslant G_2$. Repeating the same argument, $|H \cap G_1| = p_2^{l_2}$. By comparing the indices, $G = (H \cap G_1)G_2$. Define $M^G := \langle M^g \rangle$. Then $M^G = M^{G_2} \leqslant G_2 < G$ and therefore M^G is a proper normal subgroup of G. Set $N := M^G$.

Let K be a Hall p'-subgroup of G for some $p \in \{p_1, p_2, \dots, p_k\}$. Then $K \cap N$ is a Hall p'-subgroup of N and KN/N is a Hall p'-subgroup of G/N. By minimality of G, we derive that N and G/N are solvable. Thus G is solvable.

□

The set of G_i, which are Hall p_i'-subgroups of G, that we introduced in the beginning of this proof is known as a *Sylow system* of G.

As we discussed before, as a consequence of Theorem 4.14 and Theorem 3.5 we have the following result.

Corollary 4.17. *Let G be a finite group. G is solvable if and only if it has Hall π-subgroups for every set of primes π. Moreover, all Hall π-subgroups of G are conjugate to each other.*

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