Chapter 7

# Confidence Intervals

# Confidence Intervals

- In the previous chapter, we discussed situations in which the population parameter is known.
- But in reality, we usually know the sample statistics and use them to estimate the population parameter.

# Confidence Intervals

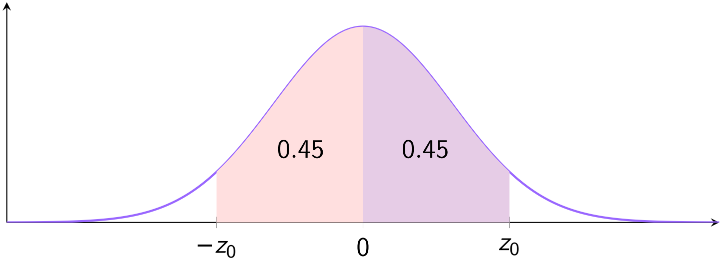
- The unknown population parameter (for example, a mean or a proportion) that we want to estimate is called the target parameter.
- If the sample mean is $\bar{x}$, then our estimate of the population mean is also $\bar{x}$. This is called a point estimate.
- More generally, we estimate an interval around $\bar{x}$, which is called a confidence interval (or interval estimate).
- This interval depends on the chosen confidence level.

# Example

For 90% confidence interval, first we find the value $z_0 > 0$ such that

$$P(-z_0 < Z < z_0) = 0.9$$

where $Z \sim N(0, 1)$.




From the standard normal table,

$$P(0 < Z < z_0) = 0.45 \quad \Longrightarrow \quad z_0 \approx 1.645.$$

Part 1

# Confidence Interval for a Population Mean:

## Large Samples ($n \geq 30$)

# Example

Suppose we know

- sample mean: $\bar{x} = 50$,
- sample size: $n = 49$,
- population std: $\sigma = 14$.

Then,

$$\text{standard error} = \frac{\sigma}{\sqrt{n}} = \frac{14}{\sqrt{49}} = 2$$

and 90% confidence interval for population mean is

$$50 \mp \underbrace{1.645 \times 2}_{3.29}$$

that is $(46.71, 53.29)$.

# Example

Suppose we know

- sample mean: $\bar{x} = 30$,
- sample size: $n = 64$,
- sample std: $s = 24$,

and we don't know the population std $\sigma$.

Then, instead of $\sigma$, we use $s$.

$$\text{standard error} = \frac{s}{\sqrt{n}} = \frac{24}{\sqrt{64}} = 3$$

and 90% confidence interval for population mean is

$$30 \mp \underbrace{1.645 \times 3}_{4.935}$$

that is $(25.065, 34.935)$.

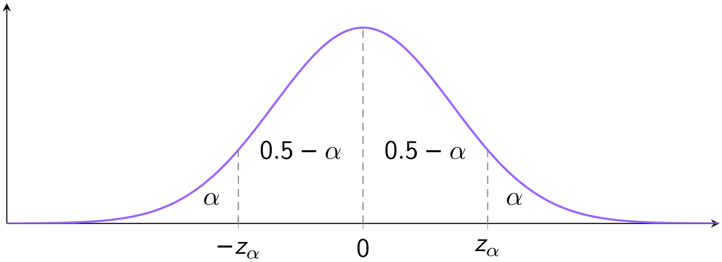
- When $\sigma$ is unknown and $n \geq 30$, we may use the approximation $\sigma \approx s$.

# Notation

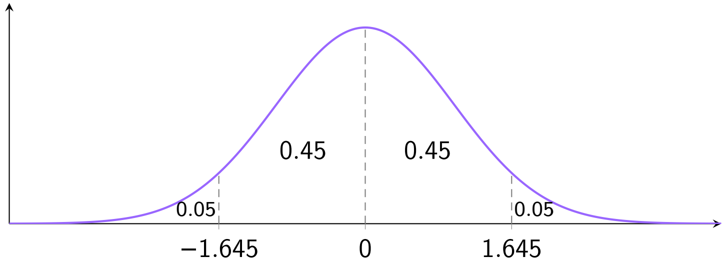
Let $\alpha$ be a real number with $0 < \alpha < 1$. The value $z_\alpha$ is defined as the number such that

$$P(Z > z_\alpha) = \alpha$$

where $Z \sim N(0, 1)$.



# Example




$$
\begin{aligned}
z_{0.05} &= 1.645 \\
z_{0.025} &= 1.96 \\
z_{0.01} &= 2.326 \\
z_{0.005} &= 2.575
\end{aligned}
$$

Since $P(Z > z_{\alpha/2}) = \frac{\alpha}{2}$, we have

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha.$$

If $n \geq 30$,then by the Central Limit Theorem,

$$P\left(\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \approx 1 - \alpha.$$

Rearranging this inequality, we get

$$P\left(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \approx 1 - \alpha.$$

# Confidence Interval for a Population Mean: Large Samples

A random sample is selected from the target population. If the sample size is $n \geq 30$,

$$100(1-\alpha)\% \text{ confidence interval for the population mean } \mu$$

is given by:

If $\sigma$ is known:

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right).$$

If $\sigma$ is unknown:

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}},\ \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right).$$

# Some Technical Terms

$$\text{confidence interval} = \text{point estimate} \mp \underbrace{z_{\alpha/2} \times \text{standard error}}_{\text{error margin}}$$

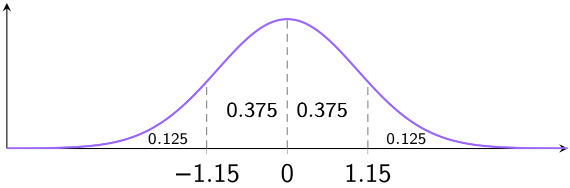
$$\text{point estimate} = \bar{x}$$

$$\text{standard error} = \frac{\sigma}{\sqrt{n}} \quad \left(\text{or } \frac{s}{\sqrt{n}}\right)$$

# Example

A random sample of 100 people is taken from a city. The average income of the sample is \$45 000 per year, with a standard deviation of \$25 000 per year. Find a 75% confidence interval for the mean yearly income of people in this city.

**Solution:**




$z_{0.125} = 1.15$. Therefore, 75% confidence interval is

$$\bar{x} \mp z_{\alpha/2}\frac{s}{\sqrt{n}} = 45\,000 \mp \underbrace{1.15 \cdot \frac{25\,000}{\sqrt{100}}}_{2\,875}$$

that is $(42\,125,\ 47\,875)$.

# Commonly Used Confidence Levels

| $100(1-\alpha)\%$ (confidence level) | $\alpha$ | $\alpha/2$ | $z_{\alpha/2}$ |
|---|---|---|---|
| 99% | 0.01 | 0.005 | 2.575 |
| 98% | 0.02 | 0.01 | 2.326 |
| 95% | 0.05 | 0.025 | 1.96 |
| 90% | 0.1 | 0.05 | 1.645 |

Going back to the last example with

$$n = 100, \quad \bar{x} = 45\,000, \quad s = 25\,000$$

the corresponding confidence intervals are

| Confidence level | $z_{\alpha/2}$ | Confidence interval |
|---|---|---|
| 99% | 2.575 | $(38\,563,\ 51\,438)$ |
| 98% | 2.326 | $(39\,185,\ 50\,815)$ |
| 95% | 1.96 | $(40\,100,\ 49\,900)$ |
| 90% | 1.645 | $(40\,888,\ 49\,113)$ |
| 75% | 1.15 | $(42\,125,\ 47\,875)$ |

- As the confidence level increases, the interval gets wider.
- There is no 100% confidence level, because the normal curve extends indefinitely in both directions. No matter how large a finite interval is, there is always some area outside it.

# Example

A bottling machine is used to fill 1-liter bottles. From long-term quality-control records, the standard deviation of the fill amount is known to be 8 mL.

A random sample of 64 bottles is taken, and the sample mean fill amount is found to be 997 mL.

Find a 95% confidence interval for the true mean fill amount.

**Solution:** Here,

$$\bar{x} = 997, \qquad \sigma = 8, \qquad n = 64.$$

For a 95% confidence interval, we use $z_{0.025} = 1.96$.

So the 95% confidence interval is

$$\bar{x} \mp z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 997 \mp 1.96 \cdot \frac{8}{\sqrt{64}} = 997 \mp 1.96.$$

Therefore, the confidence interval is

$$(995.04,\ 998.96).$$

Part 2

# Confidence Interval for a Population Proportion:

## Large Samples ($n\hat{p} \geq 15$ and $n(1 - \hat{p}) \geq 15$)

# Example

A public opinion poll is conducted in the U.S. to estimate the proportion of citizens who trust the president.

Suppose that 1 000 people are randomly selected, and 397 say that they trust the president.

Find a 95% confidence interval for the proportion of U.S. citizens who trust the president.

- Here, the sample proportion is

$$\hat{p} = \frac{\text{Number of sampled people who trust the president}}{\text{Number of people sampled}} = \frac{397}{1\,000} = 0.397$$

- We want to estimate the population proportion

$$p = \frac{\text{Number of U.S. citizens who trust the president}}{\text{Number of U.S. citizens}}$$

We proceed similarly, this time using the normal approximation to the binomial distribution.

Recall that

$$\sigma = \sqrt{p(1-p)}, \qquad s = \sqrt{\hat{p}(1-\hat{p})}.$$

But here we do not know $\sigma$. (In fact, knowing $\sigma$ would almost determine $p$.)

Therefore, for a large sample, we use $s$ instead of $\sigma$.

Hence, the standard error is

$$\frac{s}{\sqrt{n}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

# Confidence Interval for a Population Proportion: Large Samples

A random sample is selected from the target population. Let $n$ be the sample size and $\hat{p}$ be the sample proportion. If $n\hat{p} \geq 15$ and $n(1-\hat{p}) \geq 15$,

$$100(1-\alpha)\% \text{ confidence interval for the population proportion } p$$

is given by:

$$\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right).$$

# Example (cont'd)

A public opinion poll is conducted in the U.S. to estimate the proportion of citizens who trust the president. Suppose that 1 000 people are randomly selected, and 397 say that they trust the president. Find a 95% confidence interval for the proportion of U.S. citizens who trust the president.

**Solution:** Here,

$$\hat{p} = 0.397, \qquad n = 1\,000.$$

First of all,

$$n\hat{p} = 397 > 15, \quad n(1-\hat{p}) = 1\,000 - 397 = 603 > 15.$$

For a 95% confidence interval, we use $z_{0.025} = 1.96$.

So the 95% confidence interval is

$$\hat{p} \mp z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.397 \mp 1.96\sqrt{\frac{0.397 \cdot 0.603}{1\,000}} \approx 0.397 \mp 0.0303.$$

Therefore, the confidence interval is

$$(0.3667,\ 0.4273) = (36.67\%,\ 42.73\%).$$

Part 3

# Confidence Interval for a Population Mean:

## Small Samples ($n < 30$)

## with a Normal Population

# If $\sigma$ is Known

A random sample of size $n < 30$ is selected from a population that is approximately normal.

If the population standard deviation $\sigma$ is known,

$$100(1-\alpha)\% \text{ confidence interval for the population mean } \mu$$

is given by

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right).$$

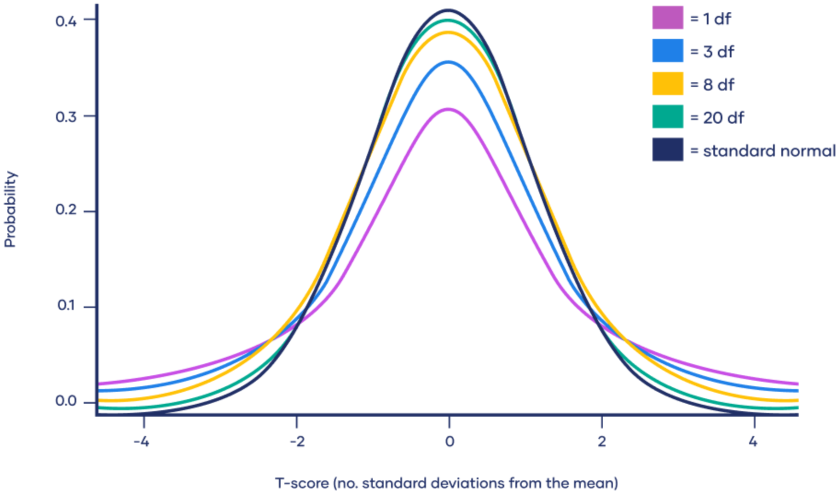
# If $\sigma$ is Unknown

What if $\sigma$ is unknown?

Since $n < 30$, the approximation $\sigma \approx s$ is not reliable enough for our purposes.

In that case, we use another curve, called the $t$-curve, instead of the normal curve.

# $t$-curve



= 1 df
= 3 df
= 8 df
= 20 df
= standard normal
Probability
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4
T-score (no. standard deviations from the mean)

# $t$-curve

There is only one standard normal curve, but there are infinitely many different $t$-curves.

$$\text{degrees of freedom} = \text{sample size} - 1$$

Each value of the degrees of freedom gives a different $t$-curve.
(We will later study two-sample problems, where the degrees of freedom are calculated differently.)

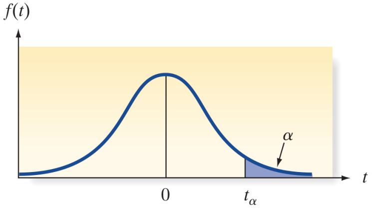
Each $t$-curve is symmetric about $t = 0$, just like the standard normal curve.

# $t$-table

Let $\alpha$ be a real number with $0 < \alpha < 1$. For a given number of degrees of freedom, the value $t_\alpha$ is defined as the number such that

$$P(T > t_\alpha) = \alpha,$$

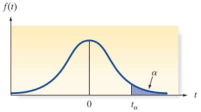
where $T$ has the corresponding $t$-distribution.




We use a $t$-table to find values of $t_\alpha$ for common values of $\alpha$.

Each row gives the values for a different degrees of freedom.

# $t$-table




| Degrees of Freedom | $t_{.100}$ | $t_{.050}$ | $t_{.025}$ | $t_{.010}$ | $t_{.005}$ | $t_{.001}$ | $t_{.0005}$ |
|---|---|---|---|---|---|---|---|
| **1** | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 318.31 | 636.62 |
| **2** | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 22.326 | 31.598 |
| **3** | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 10.213 | 12.924 |
| **4** | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 7.173 | 8.610 |
| **5** | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 | 6.869 |
| **6** | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
| **7** | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.785 | 5.408 |
| **8** | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 | 5.041 |
| **9** | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| **10** | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 | 4.587 |
| **11** | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
| **12** | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.930 | 4.318 |
| **13** | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.852 | 4.221 |
| **14** | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.787 | 4.140 |
| **15** | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.733 | 4.073 |
| **16** | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.686 | 4.015 |
| **17** | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.646 | 3.965 |
| **18** | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.610 | 3.922 |
| **19** | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.579 | 3.883 |
| **20** | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.552 | 3.850 |
| **21** | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.527 | 3.819 |
| **22** | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.505 | 3.792 |
| **23** | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.485 | 3.767 |
| **24** | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.467 | 3.745 |
| **25** | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.450 | 3.725 |
| **26** | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.435 | 3.707 |
| **27** | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.421 | 3.690 |
| **28** | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.408 | 3.674 |
| **29** | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.396 | 3.659 |
| **30** | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.385 | 3.646 |
| **40** | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 3.307 | 3.551 |
| **60** | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 3.232 | 3.460 |
| **120** | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 3.160 | 3.373 |
| **∞** | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 3.090 | 3.291 |

# If $\sigma$ is Unknown

A random sample of size $n < 30$ is selected from a population that is approximately normal.

If the population standard deviation $\sigma$ is unknown,

$100(1-\alpha)\%$ confidence interval for the population mean $\mu$

is given by

$$\left(\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}},\ \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

where $t_{\alpha/2}$ is taken with $n-1$ degrees of freedom.

# Confidence Interval for a Population Mean: Small Samples

A random sample is selected from a population that is approximately normal. If the sample size is $n < 30$,

$$100(1-\alpha)\% \text{ confidence interval for the population mean } \mu$$

is given by:

If $\sigma$ is known:

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right).$$

If $\sigma$ is unknown:

$$\left(\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}},\ \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

where $t_{\alpha/2}$ is taken with $n-1$ degrees of freedom.

# Example

The lifetime of a certain type of battery is approximately normally distributed.

A random sample of 12 batteries is tested. The sample mean lifetime is 52.4 hours, and the sample standard deviation is 4.6 hours.

Find a 95% confidence interval for the true mean lifetime of this type of battery.

**Solution:** Here,

$$n = 12, \qquad \bar{x} = 52.4, \qquad s = 4.6.$$

Since $\sigma$ is unknown and $n < 30$, we use a $t$-curve with

$$n - 1 = 11$$

degrees of freedom.

# Example (cont'd)

For a 95% confidence interval, we use

$$t_{0.025} = 2.201.$$

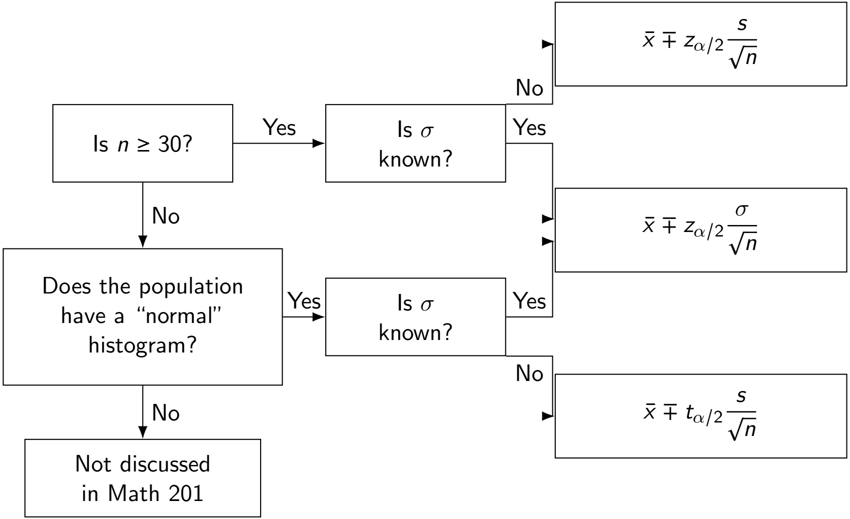
So the confidence interval is

$$\begin{aligned} \bar{x} \mp t_{\alpha/2}\frac{s}{\sqrt{n}} &= 52.4 \mp 2.201 \cdot \frac{4.6}{\sqrt{12}} \\ &\approx 52.4 \mp 2.92. \end{aligned}$$

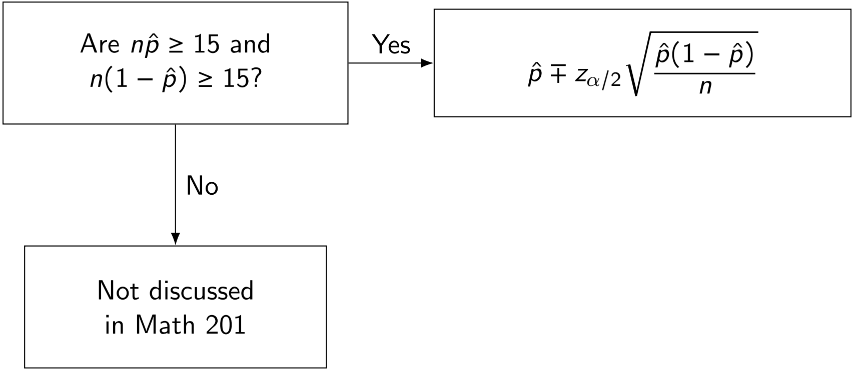
Therefore, the confidence interval is

$$(49.48,\ 55.32).$$

# Confidence Interval Flow Chart



# Confidence Interval Flow Chart for a Population Proportion



Part 4

# Determining the Sample Size

# Example

A random sample of cats is taken from İstanbul. Based on this sample, an 80% confidence interval for the mean weight of cats in İstanbul is found to be (3.744 kg, 4.256 kg). If the population standard deviation of cat weights is 1.6 kg, what are the sample mean and the sample size?

**Solution:**

The sample mean is the midpoint of the confidence interval:

$$\bar{x} = \frac{3.744 + 4.256}{2} = 4.00.$$

The margin of error is

$$E = 4.256 - 4.00 = 0.256.$$

For an 80% confidence interval, we have $z_{0.10} \approx 1.28$.

# Example (cont'd)

Using

$$E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

we get

$$0.256 = 1.28 \cdot \frac{1.6}{\sqrt{n}}.$$

So

$$\sqrt{n} = \frac{1.28 \cdot 1.6}{0.256} = 8$$

and hence

$$n = 8^2 = 64.$$

# Determining the Sample Size (for $\mu$)

**Aim:** Estimate $\mu$ with an error margin at most $d$.

If

$$d \geq \text{error margin} = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

then

$$n = \frac{\left(z_{\alpha/2}\right)^2\sigma^2}{\text{error margin}^2} \geq \frac{\left(z_{\alpha/2}\right)^2\sigma^2}{d^2}$$

# Determining the Sample Size (for $\mu$)

**Result:** In order to estimate $\mu$ with an error margin of at most $d$ and $100(1-\alpha)\%$ confidence level, the required minimum sample size is

$$\frac{\left(z_{\alpha/2}\right)^2 \sigma^2}{d^2}.$$

(Compute this value and round UP to the next integer.)

The value of $\sigma$ is usually unknown. It can be estimated by the standard deviation $s$ from a previous sample.

$$\frac{\left(z_{\alpha/2}\right)^2 s^2}{d^2}.$$

# Example

A factory wants to estimate the mean diameter of a certain type of metal rod.

They want the estimate to have an error margin at most 0.02 mm, with a 99% confidence level.

A previous study suggests that the standard deviation is about 0.08 mm.

What sample size is required?

**Solution:**

Here,

$$d = 0.02, \qquad z_{0.005} = 2.575, \qquad s = 0.08.$$

So

$$n \geq \frac{(z_{\alpha/2})^2 s^2}{d^2} = \frac{(2.575)^2 (0.08)^2}{(0.02)^2} = 106.09$$

Rounding up to the next integer, we get

$$n = 107.$$

## Determining the Sample Size (for $p$)

For proportions, we have:

$$\sigma^2 = p(1-p) \approx \hat{p}(1-\hat{p})$$

(Because the value of $p$ is unknown, it can be estimated by the sample proportion $\hat{p}$ from a previous sample.)

Substituting this into our formula, we get

$$\frac{\left(z_{\alpha/2}\right)^2 \sigma^2}{d^2} \approx \frac{\left(z_{\alpha/2}\right)^2 \hat{p}(1-\hat{p})}{d^2}$$

**Result:** In order to estimate $p$ with an error margin of at most $d$ and $100(1-\alpha)\%$ confidence level, the required minimum sample size is

$$\frac{\left(z_{\alpha/2}\right)^2 \hat{p}(1-\hat{p})}{d^2}.$$

(Compute this value and round UP to the next integer.)

# Example

A university wants to estimate the proportion of its students who use public transportation to come to campus.

They want the estimate to have an error margin of at most 0.03, with a 95% confidence level.

A previous survey suggests that the proportion is about 0.60.

What sample size is required?

**Solution:**

Here,

$$d = 0.03, \qquad z_{0.025} = 1.96, \qquad \hat{p} = 0.60.$$

Then

$$n \geq \frac{(z_{\alpha/2})^2 \hat{p}(1-\hat{p})}{d^2} = \frac{(1.96)^2(0.60)(0.40)}{(0.03)^2} \approx 1024.43.$$

Rounding up to the next integer, we get $n = 1025$.

# Determining the Sample Size (for $p$)

The maximum value of $p(1-p)$ is $1/4$. So for any $p$,

$$\frac{\left(z_{\alpha/2}\right)^2}{4d^2} \geq \frac{\left(z_{\alpha/2}\right)^2 p(1-p)}{d^2}$$

Hence, taking

$$n \geq \frac{\left(z_{\alpha/2}\right)^2}{4d^2}$$

guarantees that the error margin is at most $d$.

If we have no prior estimate of $p$, we will use this formula.

# Determining the Sample Size (for $p$)

**Result:** If we have no prior estimate of $p$, then in order to estimate $p$ with of an error margin at most $d$ and $100(1-\alpha)\%$ confidence level, the required minimum sample size is

$$\frac{\left(z_{\alpha/2}\right)^2}{4d^2}.$$

(Compute this value and round UP to the next integer.)

## Example

A campaign team wants to estimate the proportion of voters who support their candidate.

They want the estimate to have an error margin of at most 0.03, with a 95% confidence level.

What sample size is required?

**Solution:**

Here,

$$d = 0.03, \qquad z_{0.025} = 1.96.$$

Then

$$n \geq \frac{(1.96)^2}{4 \cdot (0.03)^2} \approx 1067.11.$$

Rounding up to the next integer, we get

$$n = 1068.$$