Chapter 5 - Part 1

# Continuous Random Variables

# Probability Distribution Function - Definition

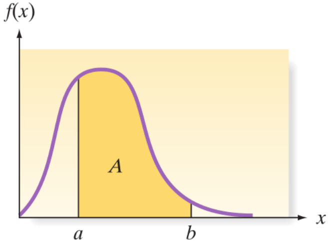
The probability distribution function (PDF) (or probability density function) of a continuous random variable is a non-negative function that can be represented by a piecewise smooth curve such that the area under this curve is 1.

- **Meaning:** The probability distribution function of a continuous random variable $X$ is a function $f$ such that
  - $f(x) \geq 0$ for all possible values $x$ of $X$,
  - the graph of $f(x)$ is "nice enough" so that we can talk about the area under it,
  - the total area under the graph of $f(x)$ is 1.

# Probability Distribution Function

The areas under the PDF correspond to probabilities.

Let $X$ be a continuous random variable with the following PDF.




$$P(a < X < b) = \text{ the area } A \text{ under the curve between } a \text{ and } b$$

Because there is no area over a single point, say, $X = a$, it follows that

$$P(X = a) = 0$$

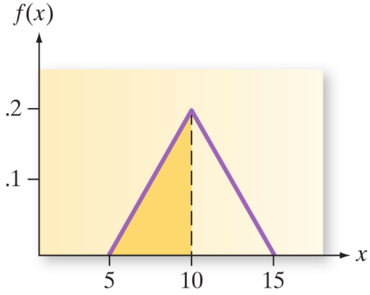
and

$$P(a \leq X \leq b) = P(a < X < b)$$

In other words, the probability is the same regardless of whether you include the endpoints of the interval.

# Example

Let $X$ be a continuous random variable with the following PDF.

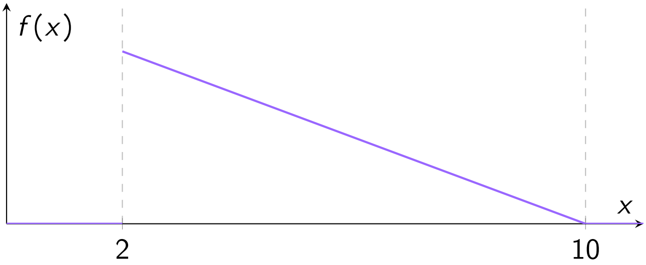



Find $P(5 < X < 10)$.

**Solution:** $P(5 < X < 10) = \frac{5 \times 0.2}{2} = 0.5.$

# Example

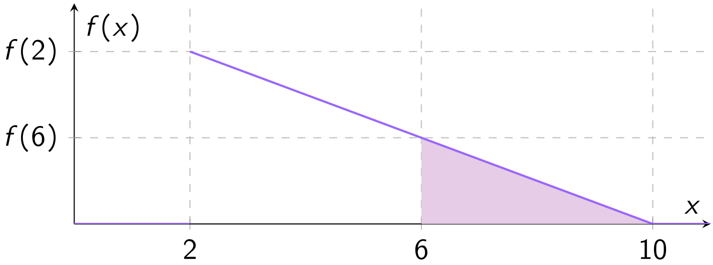
Let $X$ be a continuous random variable with the following PDF.




Find $P(6 < X)$.

# Example - Solution

$P(6 < X)$ is the shaded area.



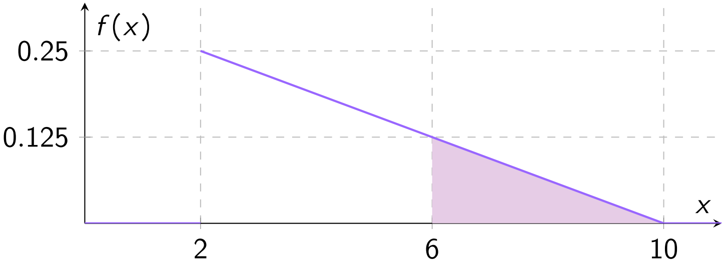

We should have

$$\frac{8 \times f(2)}{2} = 1 \implies f(2) = \frac{1}{4} = 0.25$$

Then,

$$\frac{f(6)}{4} = \frac{f(2)}{8} \implies f(6) = \frac{f(2)}{2} = 0.125$$

# Example - Solution (cont'd)




Thus

$$P(6 < X) = P(6 < X < 10) = \frac{4 \times 0.125}{2} = 0.25$$

# Percentiles

Let $X$ be a continuous random variable with the PDF $f(x)$.

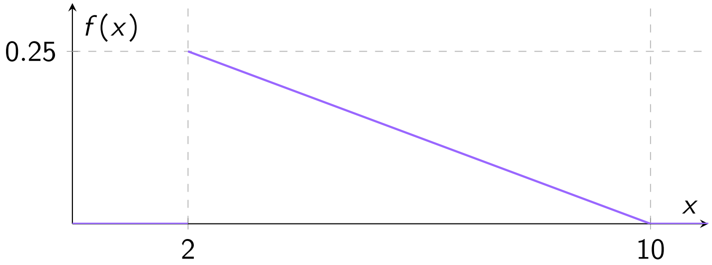
Let $0 \leq k \leq 100$. The $k^{\text{th}}$ percentile of $X$ is a number $n$ such that

$$P(X < n) = \frac{k}{100} = k\%.$$

$50^{\text{th}}$ percentile is called the median.

# Example

Let $X$ be a continuous random variable with the following PDF.

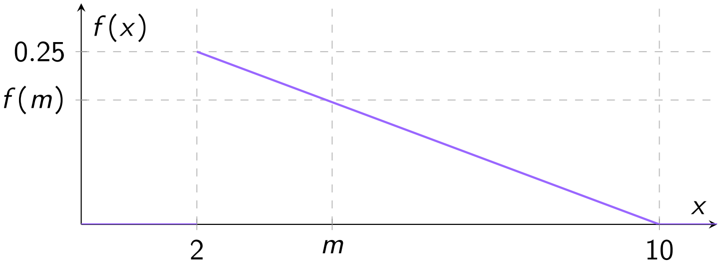



Find the median of $X$.

# Example - Solution

We are looking for $m$ such that

$$P(2 < X < m) = P(m < X < 10) = 0.5.$$

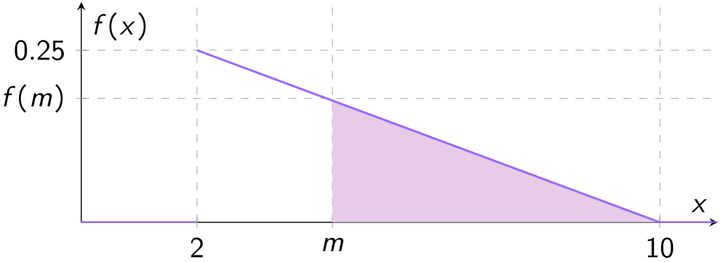



First,

$$\frac{f(m)}{10-m} = \frac{0.25}{8} \implies 32f(m) = 10 - m \implies f(m) = \frac{10-m}{32}$$

# Example - Solution (cont'd)

$$P(2 < X < m) = P(m < X < 10) = 0.5.$$




Also,

$$\frac{f(m) \times (10 - m)}{2} = 0.5 \implies f(m) \times (10 - m) = 1$$

# Example - Solution (cont'd)

Thus

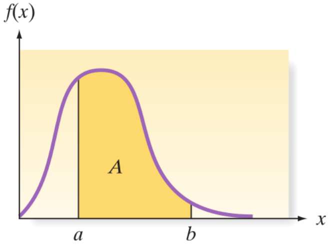
$$
\begin{aligned}
&\frac{10-m}{32} \times (10-m) = 1 \\
\Longrightarrow\ & (10-m)^2 = 32 \\
\Longrightarrow\ & 10-m = \mp\sqrt{32} \\
\Longrightarrow\ & 10-m = \sqrt{32} \quad (\text{since } 10-m>0) \\
\Longrightarrow\ & m = 10-\sqrt{32} \approx 4.34.
\end{aligned}
$$

# Integral Sign

We will use the notation

$$A = \int_a^b f(x)\,dx$$

to denote the area under the graph of $f(x)$ between $x = a$ and $x = b$.




**Remark:** As it is not included in the scope of this course, you will not calculate any integral!

# Integral Sign

Let $X$ be a continuous random variable with the PDF $f(x)$. Then,

$$P(a < X < b) = \int_a^b f(x)\,dx$$

$$P(a < X) = \int_a^\infty f(x)\,dx$$

$$P(X < b) = \int_{-\infty}^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = 1$$

# Expected Value and Variance

Let $X$ be a continuous random variable with the PDF $f(x)$.

The expected value (or mean) of $X$ is

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

The variance of $X$ is

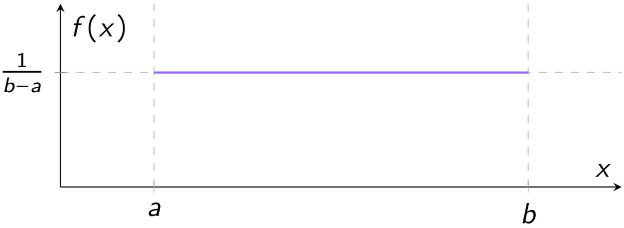
$$\sigma^2 = \mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = E(X^2) - E(X)^2$$

The standard deviation of $X$ is the square root of its variance.

$$\mathrm{StDev}(X) = \sigma = \sqrt{\sigma^2}$$

# The Uniform Distribution

A continuous random variable $X$ is said to have a uniform distribution if it has a PDF of the following form for some numbers $a < b$.




$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise} \end{cases}$$

In that case, we denote

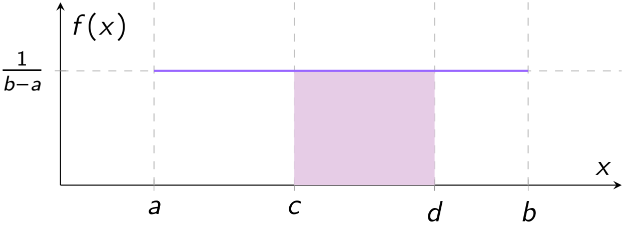
$$X \sim U(a, b).$$

# Proposition

Let $X \sim U(a, b)$ and $a \leq c < d \leq b$. Then

$$P(c < X < d) = \frac{d - c}{b - a}$$

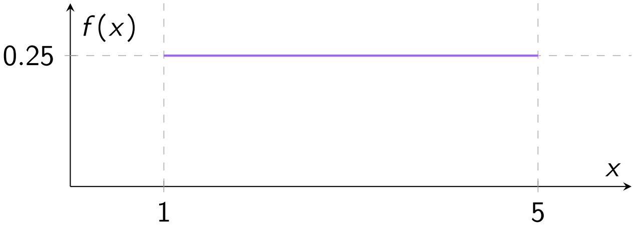
*Proof.*




□

# Example

Let $X \sim U(1, 5)$.

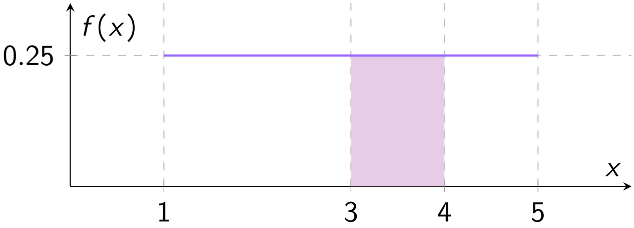



$$\frac{1}{5-1} = 0.25$$

# Example

Let $X \sim U(1,5)$.

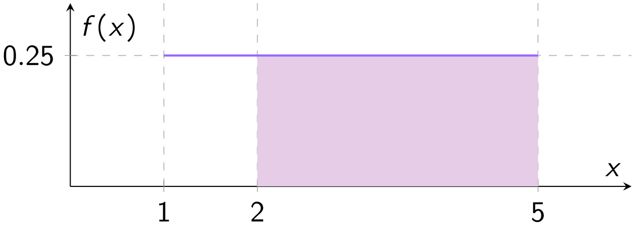
$$P(3 < X < 4) = \frac{4-3}{5-1} = \frac{1}{4} = 0.25$$



# Example

Let $X \sim U(1,5)$.

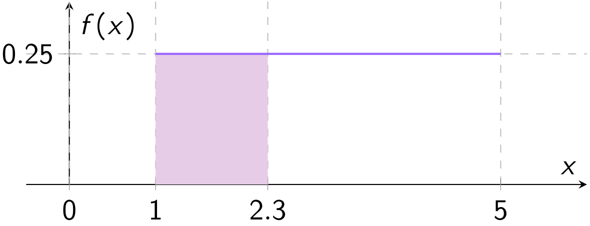
$$P(2 < X) = P(2 < X < 5) = \frac{5-2}{5-1} = \frac{3}{4} = 0.75$$



# Example

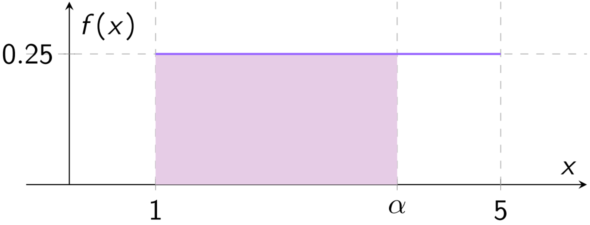
Let $X \sim U(1, 5)$.

$$P(0 < X < 2.3) = P(1 < X < 2.3) = \frac{2.3 - 1}{5 - 1} = \frac{1.3}{4} = 0.325$$



# Example

Let $X \sim U(1,5)$. Find the $70^{\text{th}}$ percentile of $X$.




We are looking for $\alpha$ such that

$$P(1 < X < \alpha) = 0.7.$$

Then,

$$\frac{\alpha - 1}{5 - 1} = 0.7 \implies \frac{\alpha - 1}{4} = 0.7 \implies \alpha - 1 = 2.8 \implies \alpha = 3.8$$

# Proposition

Let $X \sim U(a, b)$. Then

$$
\begin{aligned}
E(X) &= \frac{a+b}{2} \\
\text{median of } X &= \frac{a+b}{2} \\
\mathrm{Var}(X) &= \frac{(b-a)^2}{12} \\
\mathrm{StDev}(X) &= \frac{b-a}{\sqrt{12}}
\end{aligned}
$$

# Example

Let $X \sim U(1, 5)$. Then

$$\begin{aligned}
E(X) &= \frac{1+5}{2} = 3 \\
\text{median of } X &= \frac{1+5}{2} = 3 \\
\text{Var}(X) &= \frac{(5-1)^2}{12} = \frac{16}{12} = \frac{4}{3} \approx 1.33 \\
\text{StDev}(X) &= \sqrt{\frac{4}{3}} \approx 1.15
\end{aligned}$$

# Example

Let $X \sim U(-1, 4)$. Then

$$E(X) = \frac{-1+4}{2} = 1.5$$

$$\text{median of } X = \frac{-1+4}{2} = 1.5$$

$$\mathrm{Var}(X) = \frac{(4-(-1))^2}{12} = \frac{25}{12} \approx 2.08$$

$$\mathrm{StDev}(X) = \sqrt{\frac{25}{12}} \approx 1.44$$

# Proof (Calculus Required!)

Let $X \sim U(a, b)$. Then $E(X) = \frac{a+b}{2}$.

*Proof.*

$$E(X) = \int_a^b \frac{1}{b-a} x \, dx = \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.$$

$\square$

## Proof (Calculus Required!)

Let $X \sim U(a, b)$. Then $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$

Proof.

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\
&= \int_a^b \frac{1}{b-a} x^2 \, dx - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{b-a} \frac{x^3}{3}\Big|_a^b - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{(b-a)^2}{12}.
\end{aligned}$$

$\square$