Chapter 3 - Part 3

Conditional Probability

Motivating Example

	Smoker: Yes	Smoker: No	Total
Male	18	42	60
Female	10	30	40
Total	28	72	100

$$P(\text{ randomly selected individual is a smoker }) = \frac{28}{100}$$

$$P(\text{ randomly selected individual is a female }) = \frac{40}{100}$$

$$P(\text{ randomly selected individual is a female } \textbf{and} \text{ a smoker }) = \frac{10}{100}$$

$$P(\text{ randomly selected individual is a female } \textbf{or} \text{ a smoker }) = \frac{28}{100} + \frac{40}{100} - \frac{10}{100} = \frac{58}{100}$$

Motivating Example

	Smoker: Yes	Smoker: No	Total
Male	18	42	60
Female	10	30	40
Total	28	72	100

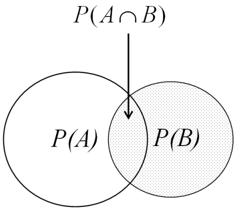
$$P(\text{ randomly selected female is a smoker }) = \frac{10}{40}$$

$$P(\text{ randomly selected smoker is a female }) = \frac{10}{28}$$

Definition

- Let A and B be events in a sample space S with $P(B) \neq 0$.
- The conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$



- **Interpretation:** Given that B has occurred, what is the probability that A occurs?

Example

- Roll a fair die.
- Given that the outcome is **odd**, what is the probability of rolling a **1**?

- Let
  - Event B: the outcome is odd $\rightarrow B = \{1, 3, 5\}$
  - Event A: the outcome is 1 $\rightarrow A = \{1\}$
  - $\implies A \cap B = \{1\}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}.$$

Example

- A factory tests cloth strips for defects in **length** and **texture**.
  - 10% failed the length test
  - 5% failed the texture test
  - 0.8% failed both test
- If a strip failed the length test, what is the probability that it also fails the texture test?

- Given

$$P(L) = 0.1, \quad P(T) = 0.05, \quad P(L \cap T) = 0.008.$$

we have

$$P(T \mid L) = \frac{P(L \cap T)}{P(L)} = \frac{0.008}{0.1} = 0.08.$$

Example

Your friend has two children. One is a boy. What is the probability that the other child is also a boy?

- $S = \{GG, GB, BG, BB\}$
- Event A: One child is a boy $\rightarrow A = \{GB, BG, BB\}$
- Event B: Both children are boys $\rightarrow B = \{BB\}$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Multiplication Rule

From the definition

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0)$$

we can rearrange to get the multiplication rule:

$$P(A \cap B) = P(A \mid B)\, P(B).$$

Similarly,

$$P(A \cap B) = P(B \mid A)\, P(A).$$

Example

Find the probability of drawing two aces from a standard 52-card deck if

(a) we draw cards without replacement (We don't return the first card to the deck before drawing the second card)

(b) we draw cards with replacement (We return the first card to the deck before drawing the second card)

Example

Find the probability of drawing two aces from a standard 52-card deck if

(a) we draw cards without replacement (We don't return the first card to the deck before drawing the second card)

- A: first card is an ace
- B: second card is an ace

$$P(A) = \frac{4}{52}$$

Given A, there are 51 cards left and 3 of them are aces:

$$P(B \mid A) = \frac{3}{51}$$

$$P(A \cap B) = P(A)\,P(B \mid A) = \frac{4}{52} \times \frac{3}{51}$$

Example

Find the probability of drawing two aces from a standard 52-card deck if

(b) we draw cards with replacement (We return the first card to the deck before drawing the second card)

- A: first card is an ace
- B: second card is an ace

$$P(A) = \frac{4}{52}$$

Given A, we back to the original state:

$$P(B \mid A) = P(B) = \frac{4}{52}$$

$$P(A \cap B) = P(A)\,P(B \mid A) = P(A)\,P(B) = \frac{4}{52} \times \frac{4}{52}$$

Independence

Events A and B are called independent if

$$P(A \mid B) = P(A)$$

Equivalent condition:

$$P(A \cap B) = P(A)\,P(B)$$

Another equivalent condition:

$$P(B \mid A) = P(B)$$

Otherwise they are called dependent.

- When drawing without replacement, steps are **dependent**.
- When drawing with replacement, steps are **independent**.

Example

Two coins are tossed. Are the events

- A: outcomes are the same
- B: at most one head

independent?

$$S = \{HH, HT, TH, TT\}, \quad A = \{HH, TT\}, \quad B = \{TT, HT, TH\}, \quad A \cap B = \{TT\}$$

$$P(A) = \frac{2}{4}, \quad P(B) = \frac{3}{4}, \quad P(A \cap B) = \frac{1}{4}$$

$$P(A)P(B) = \frac{2}{4} \cdot \frac{3}{4} = \frac{3}{8} \neq \frac{1}{4} = P(A \cap B)$$

So A and B are **dependent**.

Example

Three coins are tossed. Are the events

- A: outcomes are the same
- B: at most one head

independent?

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

$$A = \{HHH, TTT\}, \quad B = \{HTT, THT, TTH, TTT\}, \quad A \cap B = \{TTT\}$$

$$P(A) = \frac{2}{8}, \quad P(B) = \frac{4}{8}, \quad P(A \cap B) = \frac{1}{8}$$

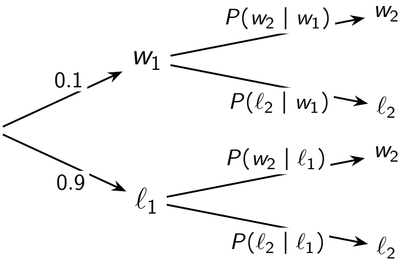
$$P(A)P(B) = \frac{2}{8} \cdot \frac{4}{8} = \frac{1}{8} = P(A \cap B)$$

So A and B are **independent**.

Example

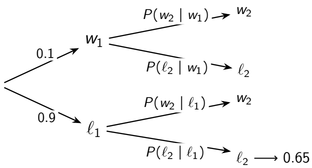
A basketball team will win the next game with probabilty 0.1, and the game after that with probabilty 0.3. The chance of losing both games is 0.65. What is the probabilty that the team will win exactly one of these two games?

$$S = \{w_1 \cap w_2, w_1 \cap \ell_2, \ell_1 \cap w_2, \ell_1 \cap \ell_2\}$$



$$P(\ell_1 \cap \ell_2) = 0.65, \quad P(w_1 \cap w_2) + P(\ell_1 \cap w_2) = 0.3$$

Example (cont'd)



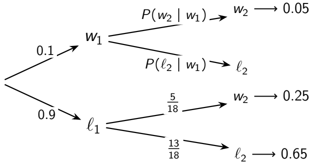
$$P(\ell_2 \mid \ell_1) = \frac{P(\ell_1 \cap \ell_2)}{P(\ell_1)} = \frac{0.65}{0.9} = \frac{13}{18}$$

$$P(w_2 \mid \ell_1) = 1 - P(\ell_2 \mid \ell_1) = 1 - \frac{13}{18} = \frac{5}{18}$$

$$P(\ell_1 \cap w_2) = P(\ell_1)P(w_2 \mid \ell_1) = 0.9 \times \frac{5}{18} = 0.25$$

$$P(w_1 \cap w_2) = 0.3 - P(\ell_1 \cap w_2) = 0.3 - 0.25 = 0.05$$

Example (cont'd)

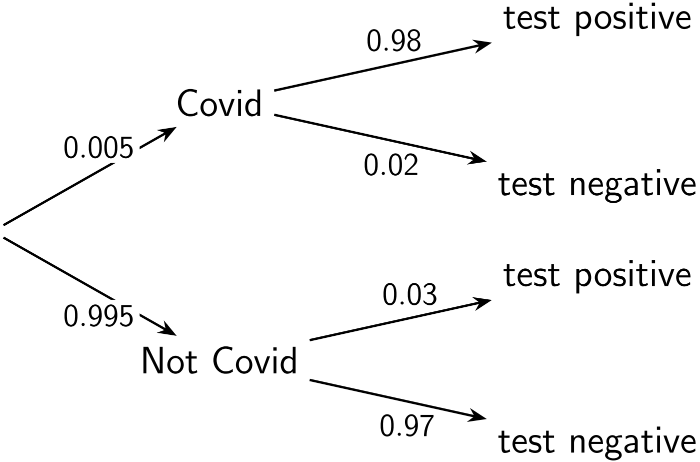


$$\underbrace{P(w_1 \cap w_2)}_{0.05} + P(w_1 \cap \ell_2) + \underbrace{P(\ell_1 \cap w_2)}_{0.25} + \underbrace{P(\ell_1 \cap \ell_2)}_{0.65} = 1$$

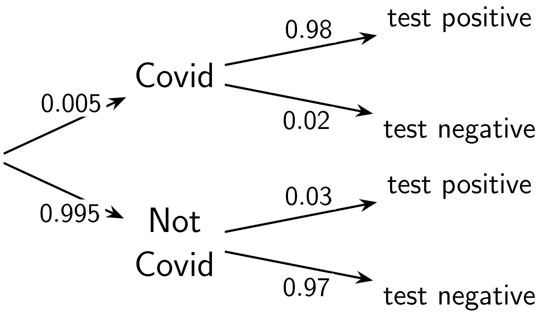
$$\implies P(w_1 \cap \ell_2) = 0.05$$

$$P(\text{ winning exactly one game }) = P(w_1 \cap \ell_2) + P(\ell_1 \cap w_2) = 0.05 + 0.25 = 0.3$$

Covid Test


Covid
Not Covid
0.005
0.995
test positive
test negative
test positive
test negative
0.98
0.02
0.03
0.97

Covid Test



Sensitivity: $P(\,TP \mid C\,) = 0.98$

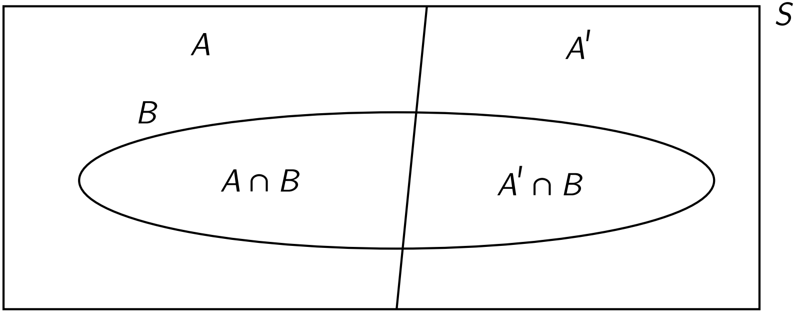
Specificity: $P(\,TN \mid C'\,) = 0.97$

False negative: $P(\,TN \mid C\,) = 0.02$

False positive: $P(\,TP \mid C'\,) = 0.03$

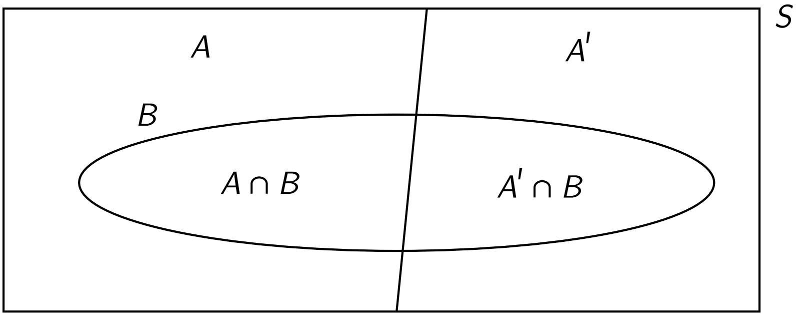
Precision: $P(\,C \mid TP\,) = ?$

Bayes' Theorem



$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}.$$

Bayes' Theorem

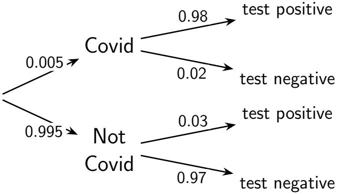


Proof.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A \cap B) + P(A' \cap B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}.$$

$\square$

Covid Test



$$\text{Sensitivity: } P(\,TP \mid C\,) = 0.98$$

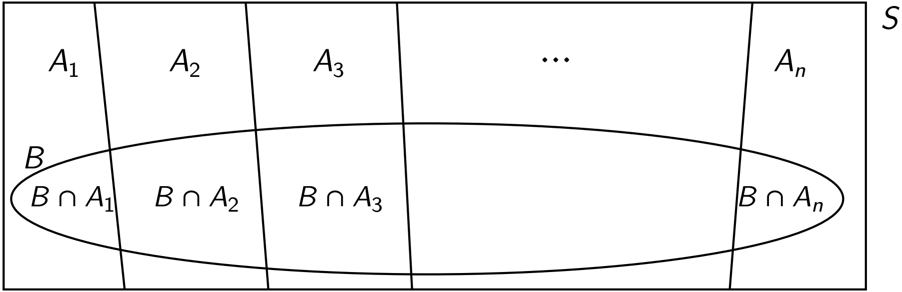
$$\text{Specificity: } P(\,TN \mid C'\,) = 0.97$$

$$\text{False negative: } P(\,TN \mid C\,) = 0.02$$

$$\text{False positive: } P(\,TP \mid C'\,) = 0.03$$

$$\begin{aligned}\text{Precision: } P(\,C \mid TP\,) &= \frac{P(\,TP \mid C\,)P(\,C\,)}{P(\,TP \mid C\,)P(\,C\,) + P(\,TP \mid C'\,)P(\,C'\,)} \\ &= \frac{0.98 \times 0.005}{0.98 \times 0.005 + 0.03 \times 0.995} \approx 0.141\end{aligned}$$

Bayes' Theorem (general case)



Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint events such that $A_1 \cup A_2 \cup \ldots \cup A_n = S$. If $P(B) \neq 0$, then

$$P(A_i \mid B) = \frac{P(B \mid A_i)\, P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)\, P(A_j)},$$

for each $i = 1, 2, \ldots, n$.

Example

A wildlife researcher is studying three bird species: Species X, Species Y, and Species Z, in a nature reserve. The proportions of these species in the reserve are as follows:

- Species X: 40%
- Species Y: 35%
- Species Z: 25%

The researcher uses a specific bird call to attract the birds, and the response rates to the call vary by species:

- If the bird is Species X, it responds to the call 80% of the time.
- If the bird is Species Y, it responds to the call 70% of the time.
- If the bird is Species Z, it responds to the call 60% of the time.

A bird responds to the call. What is the probability that the bird belongs to Species X?

Example (cont'd)

- B : bird responds to the call
- X : bird is Species X
- Y : bird is Species Y
- Z : bird is Species Z

$$
\begin{aligned}
P(X \mid B) &= \frac{P(B \mid X)P(X)}{P(B \mid X)P(X) + P(B \mid Y)P(Y) + P(B \mid Z)P(Z)} \\
&= \frac{0.8 \times 0.4}{0.8 \times 0.4 + 0.7 \times 0.35 + 0.6 \times 0.25} \\
&= \frac{0.32}{0.715} \\
&\approx 0.4476
\end{aligned}
$$