Chapter 3 - Part 2

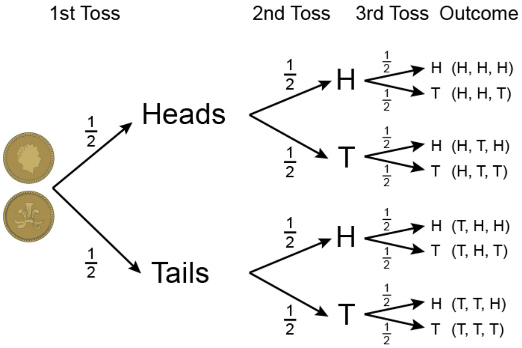
Counting

Counting

- We will discuss counting problems like
  - In how many different ways can one answer all the questions on a true/false test consisting of 20 questions?
  - How many ways can you order 7 books?
  - How many 4-letter words can be formed using the letters A, B, C, D, E, F?
  - How many ways can you put 8 pictures into 3 frames?
  - How many ways can choose 3 out of 8 pictures?

Example

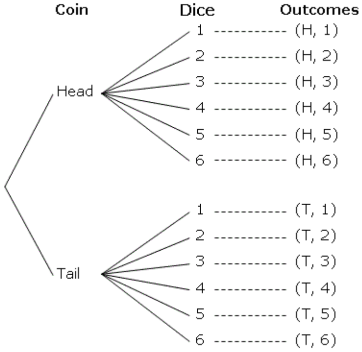
Toss a coin 3 times



Sample space has $2 \times 2 \times 2 = 8$ elements.

Example

Toss a coin, then roll a die



Sample space has $2 \times 6 = 12$ elements.

Fundamental Principle of Counting

- If there are m ways of doing a task and n ways of doing another task, then there are $m \times n$ ways of doing these tasks one after the other.
- **Example:** Roll 2 dice. How many possible outcomes are there?

$$6 \times 6 = 36$$

Example

- In how many different ways can one answer all the questions on a true/false test consisting of 20 questions?

$$\underbrace{2 \times 2 \times \ldots \times 2}_{20} = 2^{20}$$

Example

- How many ways can you order 7 books?

$$\begin{aligned}
\text{number of ways to order 7 books} &= 7 \times (\text{number of ways to order 6 books}) \\
&= 7 \times 6 \times (\text{number of ways to order 5 books}) \\
&= 7 \times 6 \times 5 \times (\text{number of ways to order 4 books}) \\
&\vdots \\
&= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1
\end{aligned}$$

Factorial

- Let n be a non-negative integer. We define the factorial of n as

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

  if $n > 0$ and we define

$$0! = 1$$

- $n!$ is the number of ways to order n objects.
- **Example:**

$$1! = 1$$
$$2! = 2$$
$$3! = 6$$
$$4! = 24$$
$$5! = 120$$

Example

- How many ways can you order 100 books?

Answer: $100! = 100 \times 99 \times \ldots \times 2 \times 1$

Example

- How many 4-letter words can be formed using the letters A, B, C, D, E, F if
  - repeats are allowed?
  - repeats are not allowed?
  - adjacent letters cannot be same?

Example

- How many 4-letter words can be formed using the letters A, B, C, D, E, F if
  - repeats are allowed?

$$6 \times 6 \times 6 \times 6 = 6^4$$

Example

- How many 4-letter words can be formed using the letters A, B, C, D, E, F if
  - repeats are not allowed?

$$6 \times 5 \times 4 \times 3 = \frac{6!}{2!}$$

Example

- How many 4-letter words can be formed using the letters A, B, C, D, E, F if
  - adjacent letters cannot be same?

$$6 \times 5 \times 5 \times 5$$

Example

- How many ways can you put 8 pictures into 3 frames?

$$8 \times 7 \times 6 = \frac{8!}{5!} = \frac{8!}{(8-3)!}$$

Permutation

- Let n be a non-negative integer and k be another integer such that $k \leq n$. We define k-permutations of n as

$$P(n,k) = \frac{n!}{(n-k)!} = n \times (n-1) \times \ldots \times (n-k+1)$$

- $P(n,k)$ is the number of ways to order n objects in k locations.
- **Example:**

$$\begin{aligned} P(n,n) &= n! \\ P(n,0) &= 1 \\ P(n,1) &= n \\ P(n,2) &= n(n-1) \\ P(n,n-1) &= n! \end{aligned}$$

Example

- How many ways can you put 50 pictures into 20 frames?

$$P(50, 20) = \frac{50!}{30!}$$

Example

- How many ways can choose 3 out of 8 pictures?

$$\underbrace{[\text{Put 8 pictures into 3 frames}]}_{P(8{,}3)\text{ many ways}} \equiv \underbrace{[\text{Choose 3 out of 8 pictures}]}_{M\text{ many ways}} \text{ then } \underbrace{[\text{Order 3 pictures}]}_{3!\text{ many ways}}$$

$$\implies P(8,3) = M \times 3!$$

$$\implies M = \frac{P(8,3)}{3!} = \frac{8!}{5! \times 3!}$$

Combination

- Let n be a non-negative integer and k be another integer such that $k \leq n$. We define k-combinations of n (a.k.a. n choose k) as

$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k! \times (n-k)!}$$

- $\binom{n}{k}$ is the number of ways to choose k out of n objects.

Combination

- **Example:**

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$\binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}$$

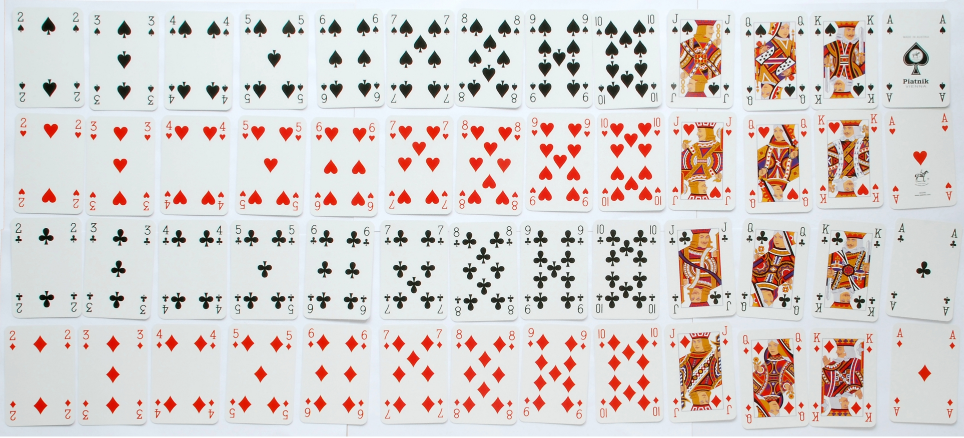
Example

- How many ways can choose 20 out of 50 pictures?

$$\binom{50}{20} = \frac{50!}{20! \times 30!}$$

Poker Hands

- In this example we will calculate the probabilities of some poker hands.
- A poker hand consists of 5 cards drawn from the deck.



♠ = spade ♥ = heart ♣ = club ♦ = diamond

Poker Hands

- What is the total possible number of hands?
- **Answer:**

$$\binom{52}{5} = 2\,598\,960$$

Poker Hands

- What is the possibility of "Four Aces"?
- **Example:**

$$\heartsuit A, \spadesuit A, \diamondsuit A, \clubsuit A, \clubsuit 5$$

or

$$\heartsuit A, \spadesuit A, \diamondsuit A, \clubsuit A, \spadesuit 7$$

or

$$\heartsuit A, \spadesuit A, \diamondsuit A, \clubsuit A, \heartsuit J$$

- **Answer:**

$$\frac{\binom{4}{4} \times \binom{48}{1}}{\binom{52}{5}} = \frac{48}{2\,598\,960} = \frac{1}{54\,145} \approx 0.0000185$$

Poker Hands

- What is the possibility of "Four of a Kind"?
- **Example:**

$$\heartsuit A, \spadesuit A, \diamondsuit A, \clubsuit A, \clubsuit 4$$

or

$$\heartsuit Q, \spadesuit Q, \diamondsuit Q, \clubsuit Q, \spadesuit 8$$

or

$$\heartsuit 5, \spadesuit 5, \diamondsuit 5, \clubsuit 5, \heartsuit 9$$

- **Answer:**

$$\frac{13 \times \binom{4}{4} \times \binom{48}{1}}{\binom{52}{5}} = \frac{624}{2\,598\,960} = \frac{1}{4\,165} \approx 0.0002401$$

Poker Hands

- What is the possibility of "Full House"?
- **Example:**

$$\heartsuit A, \spadesuit A, \diamondsuit A, \heartsuit K, \clubsuit K$$

or

$$\heartsuit Q, \diamondsuit Q, \clubsuit Q, \spadesuit 8, \clubsuit 8$$

or

$$\heartsuit 5, \spadesuit 5, \diamondsuit 5, \heartsuit 9, \clubsuit 9$$

- **Answer:**

$$\frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} = \frac{3\,744}{2\,598\,960} \approx 0.00144$$

Poker Hands

- What is the possibility of "One Pair"?

- **Example:**

$$\heartsuit A, \spadesuit A, \diamondsuit K, \heartsuit J, \clubsuit 4$$

or

$$\heartsuit K, \diamondsuit K, \clubsuit Q, \spadesuit 8, \clubsuit 2$$

or

$$\heartsuit 7, \spadesuit 7, \diamondsuit 10, \heartsuit 9, \clubsuit 3$$

- **Answer:**

$$\frac{13 \times \binom{4}{2} \times \binom{12}{3} \times \binom{4}{1} \times \binom{4}{1} \times \binom{4}{1}}{\binom{52}{5}} = \frac{1\,098\,240}{2\,598\,960} \approx 0.42257$$

Poker Hands

For the full list:
`https://en.wikipedia.org/wiki/Poker_probability`

Birthday Problem

There are 23 people in a room. What is the probability that at least two of them have the same birthday?

Solution:

- A : at least two of them have the same birthday
- A' : no two people in the room have the same birthday (all have different birthdays)

$$P(A') = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{343}{365} = \frac{P(365, 23)}{365^{23}} \approx 0.4927$$

$$\implies P(A) = 1 - P(A') = 1 - \frac{P(365, 23)}{365^{23}} \approx 0.5073$$

Birthday Problem (General Case)

There are n people in a room. What is the probability that at least two of them have the same birthday?

$$p(n) = 1 - \frac{P(365, n)}{365^n}$$

$$= 1 - \frac{365!}{365^n \times (365 - n)!}$$

n	$p(n)$
1	0.0%
5	2.7%
10	11.7%
20	41.1%
23	50.7%
30	70.6%
40	89.1%
50	97.0%
60	99.4%
70	99.9%
75	99.97%
100	99.999 97%
200	$(100 - 2 \times 10^{-28})\%$
300	$(100 - 6 \times 10^{-80})\%$
350	$(100 - 3 \times 10^{-129})\%$
365	$(100 - 1.45 \times 10^{-155})\%$
≥ 366	100%