Chapter 3 - Part 1

Fundamentals of Probability

Sample Spaces

- Experiment: A sequence of actions
- Random Experiment: An experiment whose outcome cannot be predicted
  - tossing a coin or coins
  - rolling a die or dice
  - drawing cards from a deck

Sample Spaces

- Sample Space: The set of all possible outcomes of a random experiment.
  - tossing a coin: $S = \{Head, Tail\} = \{H, T\}$
  - tossing two coins: $S = \{HH, TH, HT, TT\}$
  - toss a coin until a head occurs: $S = \{H, TH, TTH, TTTH, TTTTH, \dots\}$
  - rolling a die: $S = \{1, 2, 3, 4, 5, 6\}$
  - rolling a pair of dice:
    $S = \{(1,1), (1,2), (1,3), \dots, (6,6)\} = \{(x,y) \mid x \in \{1,2,\dots,6\}, y \in \{1,2,\dots,6\}\}$
  - drawing cards from a deck:
    $S = \{\heartsuit A, \heartsuit 2, \heartsuit 3, \dots, \heartsuit K, \spadesuit A, \spadesuit 2, \dots, \spadesuit K, \diamondsuit A, \diamondsuit 2, \dots, \diamondsuit K, \clubsuit A, \clubsuit 2, \dots, \clubsuit K\}$
  - measuring the height of a patient: $S = \{x \in \mathbb{R} \mid x > 0\}$

Sample Spaces

- Event: A subset of a sample space.
  - **Example:** Roll a die. $S = \{1, 2, 3, 4, 5, 6\}$
    - Event A: The outcome is even. $\rightarrow A = \{2, 4, 6\}$
    - Event B: The outcome is greater than 4 $\rightarrow B = \{5, 6\}$
  - **Example:** Roll a pair of dice: $S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 6)\}$
    - Event C: the sum of the outcomes is 5. $\rightarrow C = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
    - Event D: the outcomes are same $\rightarrow D = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

Probability Rules

- Let $S = \{e_1, e_2, \ldots, e_n\}$ be a sample space of some random experiment.
- The probability of e_i is a number, denoted by $P(e_i)$, satisfying the following rules:

$$0 \leq P(e_i) \leq 1, \text{ for all } i$$

and

$$P(e_1) + P(e_2) + \ldots + P(e_n) = 1.$$

Law of Large Numbers

- If an experiment is repeated a large number of times, independently under identical conditions, then the proportion of times that any specified event is expected to occur approximately equals the probability of the event's occurrence on any particular trial;
- the larger the number of repetitions, the better the approximation tends to be.

Law of Large Numbers

- Let $f(e_i)$ be the frequency of e_i after n trials. Then

$$\frac{f(e_i)}{n}$$

  is the relative frequency.

- The relative frequency $\frac{f(e_i)}{n}$ approaches the probability $P(e_i)$ as n is very large (as $n \to \infty$), i.e.,

$$P(e_i) = \lim_{n\to\infty} \frac{f(e_i)}{n}$$

Law of Large Numbers

A coin flipped n times. Frequency of heads is noted.

n	Frequency	Relative Frequency
1	0	0
10	4	0.4
100	58	0.58
1 000	505	0.505
10 000	4 980	0.498
100 000	49 916	0.49916
1 000 000	499 408	0.499408
10 000 000	4 999 377	0.4999377
100 000 000	49 998 625	0.49998625

Example

- The probability is 0.89 that the plane from City A to City B will arrive on time.
- We don't know what will happen in a particular flight. But in the long run, the proportion of arrivals will be about 0.89.

Equally Likely Outcomes

- For some random experiments, all possible outcomes are equally likely, that is all outcomes have the same probabilities:

$$P(e_1) = P(e_2) = \ldots = P(e_n) = \frac{1}{n}$$

- **Example:**
  - Tossing coins
  - Rolling dice
  - Drawing cards from a deck
  - Experiments that are equivalent to those

Monopoly

- In game *Monopoly*, you roll two dice and add the two numbers that come up (and then move by that number of squares in the game).
- Sample space:

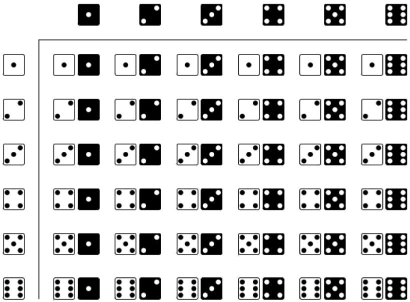
$$S = \{2, 3, 4, \ldots, 11, 12\}$$

- Are these outcomes equally likely?

Monopoly

- Consider rolling two dice (without adding). Sample space:

$$S' = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),\ldots,(6,5),(6,6)\}$$



- These outcomes are equally likely:

$$P(1,1) = P(1,2) = P(1,3) = \ldots = P(6,6) = \frac{1}{36}$$

Monopoly

$$P(\text{sum is } 2) = P(1,1) = \frac{1}{36}$$
$$P(\text{sum is } 3) = P(1,2) + P(2,1) = \frac{2}{36} = \frac{1}{18}$$
$$P(\text{sum is } 4) = P(1,3) + P(2,2) + P(3,1) = \frac{3}{36} = \frac{1}{12}$$
$$P(\text{sum is } 5) = P(1,4) + P(2,3) + P(3,2) + P(4,1) = \frac{4}{36} = \frac{1}{9}$$
$$P(\text{sum is } 6) = P(1,5) + P(2,4) + P(3,3) + P(4,2) + P(5,1) = \frac{5}{36}$$
$$P(\text{sum is } 7) = P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1) = \frac{6}{36} = \frac{1}{6}$$
$$P(\text{sum is } 8) = P(2,6) + P(3,5) + P(4,4) + P(5,3) + P(6,2) = \frac{5}{36}$$
$$P(\text{sum is } 9) = P(3,6) + P(4,5) + P(5,4) + P(6,3) = \frac{4}{36} = \frac{1}{9}$$
$$P(\text{sum is } 10) = P(4,6) + P(5,5) + P(6,4) = \frac{3}{36} = \frac{1}{12}$$
$$P(\text{sum is } 11) = P(5,6) + P(6,5) = \frac{2}{36} = \frac{1}{18}$$
$$P(\text{sum is } 12) = P(6,6) = \frac{1}{36}$$

Monopoly

- Back to the question.
- Rolling two dice and adding numbers. Sample space:

$$S = \{2, 3, 4, \ldots, 11, 12\}$$

- Outcomes are not equally likely.

Probability of an Event

- The probability of an event is the sum of the probabilities of the elements that make up this event.
- For $A = \{a_1, a_2, \ldots, a_k\} \subseteq S$,

$$P(A) = P(a_1) + P(a_2) + \ldots + P(a_k)$$

Example

- A die is loaded in such a way that each odd number is twice as likely as each even number. What is the probability of rolling a number greater than 3?

- $S = \{1, 2, 3, 4, 5, 6\}$ and

$$P(1) = P(3) = P(5) = 2k, \quad P(2) = P(4) = P(6) = k$$

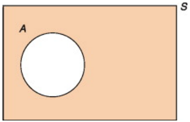
- Hence $1 = P(1) + \ldots + P(6) = 9k \implies k = \frac{1}{9}$
- The event we are looking for is $A = \{4, 5, 6\}$ and

$$P(A) = P(4) + P(5) + P(6) = k + 2k + k = 4k = \frac{4}{9}$$

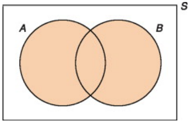
Complement, Union and Intersection

Let A and B be two events from the same sample space S.

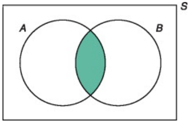
- The complement of $A \quad = \quad A' \quad = \quad S \setminus A$



- Union of A and $B \quad = \quad A$ or $B \quad = \quad A \cup B$

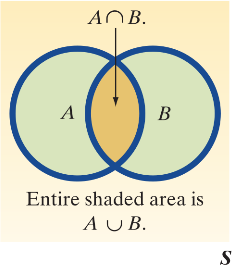


- Intersection of A and $B \quad = \quad A$ and $B \quad = \quad A \cap B$



Theorem

- $P(S) = 1$
- $P(\varnothing) = 0$
- $P(A) + P(A') = 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Example

- 2 coins are tossed:

$$S = \{HH, TH, HT, TT\}$$

- Let A be the event "at least one H":

$$A = \{HH, TH, HT\}$$

- Then, the complement of A is

$$A' = \{TT\}$$

-

$$P(A) = \frac{3}{4}, \quad P(A') = \frac{1}{4}$$

Example

- 10 coins are tossed.
- Let A be the event "at least one H". What is $P(A)$?
- $A' = \{\underbrace{TTTTTTTTTT}_{10}\}$
- Sample space has $\underbrace{2 \times 2 \times \ldots \times 2}_{10} = 2^{10} = 1024$ elements, each equally likely
- Therefore
$$P(A') = \frac{1}{1024}$$
- Hence
$$P(A) = 1 - P(A') = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

Example

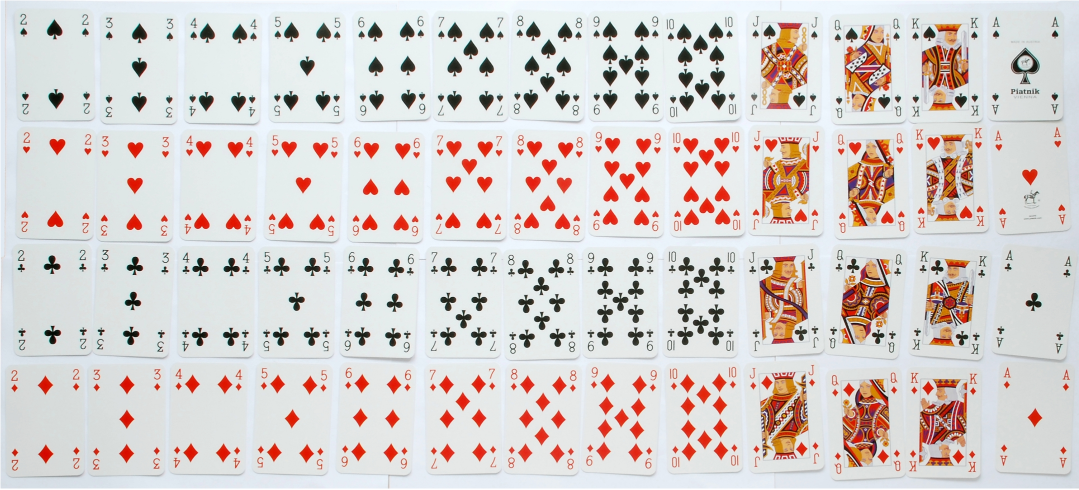
- Roll a die. $S = \{1, 2, 3, 4, 5, 6\}$
  - Event A: The outcome is even. $\to A = \{2, 4, 6\}$
  - Event B: The outcome is greater than 4 $\to B = \{5, 6\}$
- Then,
  $$A \cup B = \{2, 4, 5, 6\}, \quad A \cap B = \{6\}$$
- Hence
  $$P(A \cup B) = \frac{4}{6}$$
  and
  $$P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6}$$
  are equal.

Example

- 35.4% of population has blood type A.
- 23.2% of population has blood type $-$, "minus".
- 8.1% of population has blood type $A-$ (that is both A and minus).
- What is the probability that a randomly selected person from this population has blood type A or $-$.
- Hence

$$\begin{aligned} P(A \cup \text{minus}) &= P(A) + P(\text{minus}) - P(A \cap \text{minus}) \\ &= 0.354 + 0.232 - 0.081 = 0.505 \end{aligned}$$

Standard 52-card deck



♠ = spade ♥ = heart ♣ = club ♦ = diamond

Example

- Draw a card from a well-shuffled deck.

$$S = \{\heartsuit A, \heartsuit 2, \heartsuit 3, \ldots, \heartsuit K, \spadesuit A, \spadesuit 2, \ldots, \spadesuit K, \diamondsuit A, \diamondsuit 2, \ldots, \diamondsuit K, \clubsuit A, \clubsuit 2, \ldots, \clubsuit K\}$$

  - Event A: The card is $\heartsuit$ $\rightarrow A = \{\heartsuit A, \heartsuit 2, \heartsuit 3, \ldots, \heartsuit K\}$
  - Event B: The card is Q $\rightarrow B = \{\heartsuit Q, \spadesuit Q, \diamondsuit Q, \clubsuit Q\}$

- Then, $A \cap B = \{\heartsuit Q\}$
- Hence

$$\begin{aligned} P(\text{The card is } \heartsuit \text{ or } Q) &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13} \end{aligned}$$

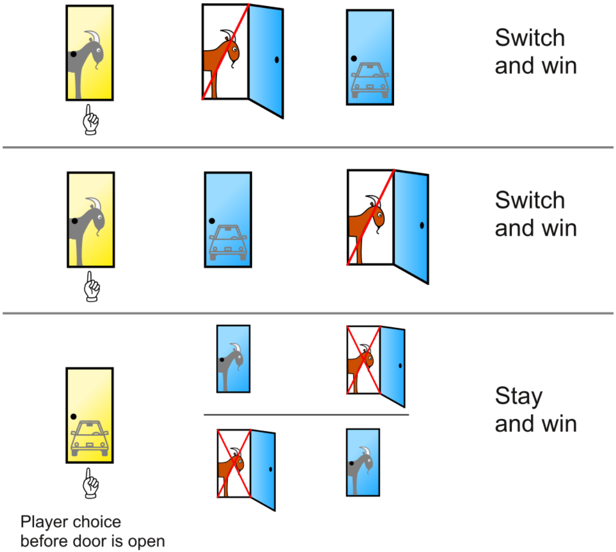
Monty Hall Problem

Suppose you're on a game show, and you're given the choice of 3 doors: Behind one door is a car; behind the others, goats.

You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.

He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Monty Hall Problem



Monty Hall Problem

Behind door 1	**Behind door 2**	**Behind door 3**	**Result if staying at door #1**	**Result if switching to the door offered**
Goat	Goat	**Car**	Wins goat	**Wins car**
Goat	**Car**	Goat	Wins goat	**Wins car**
Car	Goat	Goat	**Wins car**	Wins goat

The probability of winning the car by switching is $\frac{2}{3}$.