Algebraic Geometry

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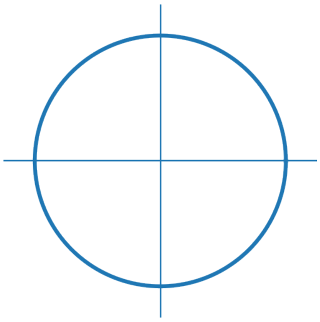
0 What is Algebraic Geometry?

At a first approximation, algebraic geometry studies geometric objects described by polynomial equations. Its basic strategy is to translate back and forth between geometry and algebra.

Take perhaps the most familiar example: the unit circle in the real plane. We can describe the same object in two different languages.

Geometry	Algebra
	$x^2 + y^2 - 1 = 0$

The picture and the equation are two descriptions of the same object. One of the central aims of algebraic geometry is to build systematic dictionaries of this kind.

In these lectures we will repeatedly move in both directions:

Geometry $\longrightarrow$ Algebra: attach a ring of functions to a geometric object.

Algebra $\longrightarrow$ Geometry: ask what kind of space is encoded by a commutative ring.

Guiding question: If polynomial equations define geometry, how much of that geometry can be recovered from the ring of functions?

Goal of the lecture

By the end of this lecture we want to regard a commutative ring not merely as an algebraic object, but as something that carries a natural geometry. The construction $\operatorname{Spec} A$ will emerge as an answer to questions raised by elementary examples.

1 From Geometry to Algebra

Consider the parabola

$$X = \{(a, b) \in k^2 : b = a^2\}.$$

At first sight, X is just a set of points satisfying one polynomial equation. But algebraic geometry asks us to look not only at the points of X, but also at the polynomial functions that live on X.

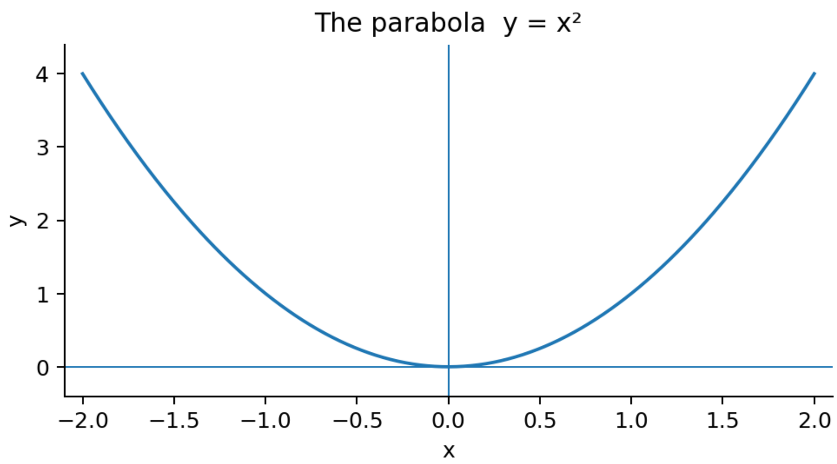


Figure 1. A familiar geometric object, described by a polynomial equation.

Every polynomial $F(x, y) \in k[x, y]$ gives a function on X by evaluation. However, different polynomials may define exactly the same function on X. For example, on X we have $y = x^2$, so y and x^2 are indistinguishable as functions on X.

More generally, two polynomials F and G give the same function on X whenever their difference vanishes on X. In this example, every multiple of $y - x^2$ vanishes on X. This leads naturally to the quotient ring

$$k[X] = k[x, y]/(y - x^2).$$

This is called the *coordinate ring* of X. It records polynomial functions on X, with two polynomials identified when they agree everywhere on X.

First principle

A polynomially defined geometric object should be studied through its ring of polynomial functions.

The parabola is algebraically the same as a line

Inside the quotient ring, the relation $y = x^2$ allows us to eliminate y. Hence

$$k[x, y]/(y - x^2) \cong k[x].$$

But $k[x]$ is also the coordinate ring of the affine line $\mathbb{A}^1_k := k^1$. This algebraic calculation reflects the familiar parametrization

$$\mathbb{A}^1_k \longrightarrow X, \qquad t \longmapsto (t, t^2),$$

whose inverse is the x-coordinate. So the line and the parabola are different subsets of the plane, but they are the same intrinsic algebraic-geometric object.

2 Reverse the direction: can a ring produce a space?

The previous example takes a geometric object X and produces a ring $k[X]$. Now reverse the question:

Guiding question

Given a commutative ring A, can we construct a geometric space from A?

Begin with $A = k[x]$. A point $a \in k$ determines an evaluation map

$$\begin{aligned} \mathrm{ev}_a : k[x] &\longrightarrow k \\ f &\longmapsto f(a) \end{aligned}$$

Its kernel is the maximal ideal $(x - a)$. Thus an ordinary point determines a maximal ideal. In multivariate, a point $(a_1, a_2, \ldots, a_n) \in k^n$ similarly gives the maximal ideal

$$(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$$

of $k[x_1, \ldots, x_n]$.

This suggests a natural question: when k is algebraically closed, does every maximal ideal of $k[x_1, \ldots, x_n]$ come from evaluation at an ordinary point? The answer is yes, and it is the form of Hilbert's Nullstellensatz that we will need.

Zariski's Lemma

Let K be a field which is finitely generated as a k-algebra. Then K is a finite algebraic extension of k.

We will use Zariski's Lemma as an algebraic black box. With it, the geometric statement we need is immediate.

Weak Nullstellensatz

Let k be algebraically closed. Every maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ is of the form

$$(x_1 - a_1, \ldots, x_n - a_n)$$

for a unique point $(a_1, \ldots, a_n) \in k^n$.

Proof. Let $\mathfrak{m}$ be maximal and set

$$K = k[x_1, \ldots, x_n]/\mathfrak{m}.$$

Since $\mathfrak{m}$ is maximal, K is a field; and by construction it is finitely generated as a k-algebra. Zariski's Lemma therefore shows that K/k is a finite algebraic extension. Since k is algebraically closed, $K = k$.

Let $a_i \in k$ be the image of x_i in K. Then $x_i - a_i \in \mathfrak{m}$ for every i, so

$$(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}.$$

The ideal on the left is maximal, hence equality holds. Uniqueness of the point is clear from the generators. ☐

Consequently, over an algebraically closed field, there is a natural bijection between

$$\text{ordinary points of affine } n\text{-space } \mathbb{A}^n_k := k^n$$

and

$$\text{the maximal ideals of } k[x_1, \ldots, x_n].$$

A first dictionary

$$\text{ordinary points} \quad \longleftrightarrow \quad \text{maximal ideals.}$$

3 Why maximal ideals are not enough

Consider now the equation

$$xy = 0.$$

Its zero set is the union of the two coordinate axes. Its coordinate ring is

$$A = k[x, y]/(xy).$$

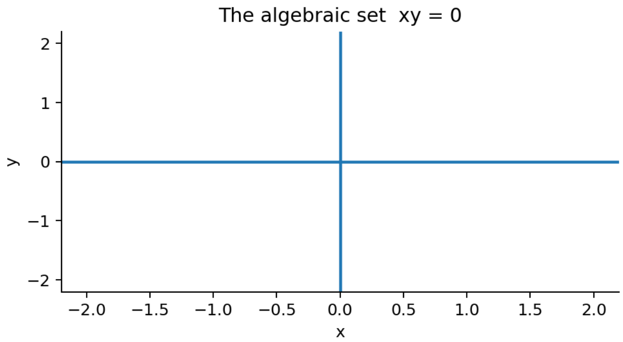


Figure 2. Two irreducible pieces meeting at the origin.

The maximal ideals of A correspond to the ordinary points on the two axes. But A contains two particularly important prime ideals:

$$(x) \qquad \text{and} \qquad (y).$$

They are not usually maximal. Geometrically, however, they have an obvious meaning: (x) corresponds to the entire y-axis, while (y) corresponds to the entire x-axis. In other words, a prime ideal can behave like a “point” representing an entire irreducible piece.

Second principle

If we use only maximal ideals, we see ordinary points. If we use all prime ideals, irreducible pieces themselves become visible as generalized points.

The spectrum

This motivates the fundamental construction

$$\operatorname{Spec} A := \{\text{prime ideals of } A\}.$$

For the moment, think of $\operatorname{Spec} A$ simply as the set of all generalized points naturally encoded by A. We will shortly give this set a topology.

Notation. From now on we write

$$\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$$

for affine n-space over k. When k is algebraically closed, the Weak Nullstellensatz says that its closed points are exactly the ordinary k-points k^n. The full spectrum also contains non-closed, or generic, points.

A structural reason for prime ideals: functoriality

There is another, deeper reason to use all prime ideals rather than only maximal ideals. A ring homomorphism naturally produces a map between spectra, in the opposite direction.

Let $\varphi : A \to B$ be a ring homomorphism. If $\mathfrak{q}$ is a prime ideal of B, then its inverse image $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of A. Therefore φ induces a map

$$\operatorname{Spec} B \longrightarrow \operatorname{Spec} A, \qquad \mathfrak{q} \longmapsto \varphi^{-1}(\mathfrak{q}).$$

The direction reverses. This is exactly what we should expect geometrically: a map of spaces pulls functions back in the opposite direction. Later, this will become the basic contravariance between affine geometry and commutative algebra.

Why not use only maximal ideals? Because inverse images of maximal ideals need not be maximal. For example, consider the inclusion

$$k[x] \longrightarrow k(x).$$

Since $k(x)$ is a field, (0) is a maximal ideal of $k(x)$. Its inverse image in $k[x]$ is again (0), but (0) is not maximal in $k[x]$. So a ring homomorphism does not in general give a map between sets of maximal ideals. Prime ideals are stable under pullback; maximal ideals are not.

Thus prime ideals are not merely extra points added for convenience: they are the class of points that makes the construction $A \mapsto \operatorname{Spec} A$ functorial.

4 The first spectra

Example: $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ when k is algebraically closed

Assume k is algebraically closed. The prime ideals of $k[x]$ are

$$(0) \qquad \text{and} \qquad (x-a) \quad (a \in k).$$

The ideals $(x-a)$ are the ordinary closed points of the affine line. The new point (0) is different: it represents the whole irreducible affine line at once. It is called the generic point.

This may look strange at first, but it is exactly the kind of point that allows irreducible subspaces to be represented inside the space itself.

A polynomial $f(x) \in k[x]$ should be regarded as a function on $\mathbb{A}^1_k$. Its value at the point $(x-a)$ is obtained by reducing modulo $(x-a)$:

$$f(x) \bmod (x-a) = f(a).$$

For example, if

$$f(x) = x^2 - 3x + 1,$$

then at the point $(x-1)$ we obtain

$$f(x) \bmod (x-1) = f(1) = -1.$$

There is also the generic point (0). At this point, the same polynomial does not specialize to a complex number: its value naturally lives in the fraction field

$$\mathbb{C}(x).$$

Thus we may think of

$$f((0)) = f(x) \in \mathbb{C}(x).$$

This suggests an important principle: the value of a function at a point of a scheme need not always lie in the ground field. It naturally lies in the residue field of that point.

We can also consider rational functions. For example,

$$g(x) = \frac{(x-3)^3}{x-2}.$$

This is not regular at the point $(x-2)$, but it is regular everywhere else. Thus, anticipating the language of the structure sheaf, g should be regarded as a regular function on

$$D(x-2) = \mathbb{A}^1_k \setminus \{(x-2)\}.$$

Moreover, its factorization tells us its local behaviour:

- g has a zero of order 3 at $(x-3)$;
- g has a pole of order 1 at $(x-2)$.

In other words, the algebraic factorization of a rational function records geometric information about where the function vanishes and where it has poles.

Example: $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$

The assumption that the ground field is algebraically closed matters. Consider $A = \mathbb{R}[x]$. Since $\mathbb{R}[x]$ is a PID, its prime ideals are (0) together with the ideals (f) generated by irreducible polynomials $f \in \mathbb{R}[x]$.

Over $\mathbb{R}$, an irreducible polynomial has degree one or two. Thus the closed points of $\operatorname{Spec} \mathbb{R}[x]$ come in two kinds:

- linear points $(x-a)$, with residue field $\mathbb{R}$;
- quadratic points (f), where f is an irreducible quadratic, with residue field $\mathbb{R}[x]/(f) \cong \mathbb{C}$.

For example, (x^2+1) is a closed point of $\operatorname{Spec} \mathbb{R}[x]$, even though x^2+1 has no real zero. Its residue field is $\mathbb{C}$. So a closed point need not be an ordinary $\mathbb{R}$-valued point when the ground field is not algebraically closed.

If we pass mentally from $\mathbb{R}$ to $\mathbb{C}$, the point (x^2+1) corresponds to the conjugate pair i and $-i$. This is a first hint that a non-rational closed point can package several geometric points together.

Example: $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ when k is algebraically closed

Now consider affine two-space $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. Here the spectrum visibly contains several geometric levels.

- The prime ideal (0) is the generic point of the whole affine plane.
- If $f \in k[x,y]$ is irreducible, the prime ideal (f) is the generic point of the irreducible plane curve $f = 0$.
- The maximal ideals $(x-a, y-b)$ are the ordinary closed points $(a,b) \in k^2$.

The inclusions between prime ideals encode incidence. If the point (a,b) lies on the irreducible curve $f = 0$, then

$$(0) \subset (f) \subset (x-a, y-b).$$

Thus one spectrum contains the plane, its irreducible curves, and its ordinary points simultaneously: each irreducible closed subspace has its own generic point.

Example: $\operatorname{Spec}\mathbb{Z}$

Now take a ring that does not look geometric at all: $A = \mathbb{Z}$. Its prime ideals are

$$(0), (2), (3), (5), (7), \ldots.$$

Thus prime numbers become points of the geometric object $\operatorname{Spec}\mathbb{Z}$. The point (0) is again generic. At the point (p), the residue field is

$$\kappa((p)) = \mathbb{Z}/(p) = \mathbb{F}_p.$$

This is one of the conceptual leaps of scheme theory: geometry is no longer restricted to subsets of k^n. A commutative ring itself can be regarded as a space, and arithmetic rings such as $\mathbb{Z}$ acquire a geometry.

An integer $n \in \mathbb{Z}$ can be regarded as a global function on $\operatorname{Spec}\mathbb{Z}$. Its value at the closed point (p) is simply its reduction modulo p:

$$n((p)) = n \bmod p \in \mathbb{F}_p.$$

For example, consider the function 100. At (3),

$$100 \bmod 3 = 1,$$

while at (2),

$$100 \bmod 2 = 0.$$

In fact,

$$100 = 2^2 \cdot 5^2,$$

so we can say more: the function 100 has a zero of order 2 at (2) and also a zero of order 2 at (5).

Now consider the rational number

$$\frac{81}{64} = \frac{3^4}{2^6} \in \mathbb{Q} = \operatorname{Frac}(\mathbb{Z}).$$

We may regard it as a rational function on $\operatorname{Spec}\mathbb{Z}$. It is regular at every closed point except (2), since its denominator is invertible modulo every prime $p \neq 2$.

Its factorization immediately gives:

- a zero of order 4 at (3);
- a pole of order 6 at (2).

At a point where the denominator is invertible, we can evaluate the function by reducing modulo that prime. For example, at (5),

$$\frac{81}{64} \equiv 81 \cdot 64^{-1} \equiv 1 \cdot 4^{-1} \equiv 1 \cdot 4 \equiv 4 \pmod 5.$$

More generally, if

$$q = \frac{a}{b} \in \mathbb{Q}^\times,$$

then the order of q at the point (p) is

$$\operatorname{ord}_{(p)}(q) = v_p(a) - v_p(b).$$

A positive value means that q has a zero at (p), a negative value means that it has a pole, and value 0 means that it is a unit near (p).

Thus $\operatorname{Spec}\mathbb{Z}$ behaves strikingly like a geometric curve: prime numbers play the role of points, and rational numbers behave like rational functions with zeros and poles at those points.

Why $\operatorname{Spec}\mathbb{Z}$ matters

It shows that the construction is not merely a new language for drawing polynomial zero sets. It extends geometric thinking to arithmetic.

Optional aside: $\operatorname{Spec}\mathbb{Q}[x]$ and Galois theory

For readers who know some Galois theory, $\operatorname{Spec}\mathbb{Q}[x]$ gives another useful picture. Its closed points are the ideals (f), where $f \in \mathbb{Q}[x]$ is irreducible. If α is a root of f, then the residue field at this point is

$$\kappa((f)) = \mathbb{Q}[x]/(f) \cong \mathbb{Q}(\alpha).$$

So a closed point of the affine line over $\mathbb{Q}$ may have a nontrivial number field as its residue field. Because $\mathbb{Q}$ has characteristic zero, f is separable; over an algebraic closure it has distinct conjugate roots, and the absolute Galois group permutes them transitively.

Informally, one closed point of $\operatorname{Spec}\mathbb{Q}[x]$ packages an entire Galois orbit of algebraic numbers. This is the arithmetic analogue of the quadratic point (x^2+1) over $\mathbb{R}$, and it hints at why schemes are a natural meeting place for geometry and number theory.

5 The Zariski topology

To make $\operatorname{Spec} A$ into a space, we must say which subsets are closed. The ring already suggests the answer. For an ideal $I \subset A$, define

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} A : I \subset \mathfrak{p}\}.$$

Interpretation: $V(I)$ is the set of generalized points at which every element of I vanishes.

Basic properties of $V(I)$

Let I and J be ideals of A, and let $\{I_\lambda\}$ be any family of ideals. Then:

$$\begin{aligned} V(0) &= \operatorname{Spec} A, \\ V((1)) &= \varnothing, \\ V(IJ) &= V(I \cap J) = V(I) \cup V(J), \\ V\left(\sum_\lambda I_\lambda\right) &= \bigcap_\lambda V(I_\lambda), \end{aligned}$$

Therefore, the sets $V(I)$ satisfy the axioms for closed sets of a topology on $\operatorname{Spec} A$. This is called the *Zariski topology*.

Moreover:

$$\begin{aligned} I \subset J &\Longrightarrow V(J) \subset V(I), \\ V(J) \subset V(I) &\Longrightarrow \sqrt{I} \subset \sqrt{J}, \\ V(I) &= V(\sqrt{I}). \end{aligned}$$

In particular, for $a, b \in A$,

$$V(ab) = V(a) \cup V(b).$$

This last identity is the set-theoretic shadow of the defining property of a prime ideal: a product vanishes at a prime point precisely when at least one factor vanishes there.

Exercise. Verify the identities above directly from the definition of $V(I)$. In particular, identify exactly where primality is used in the identity $V(ab) = V(a) \cup V(b)$.

Two useful pictures

- In $\operatorname{Spec} k[x]$, the generic point (0) lies in the closure of the whole affine line: its closure is all of $\operatorname{Spec} k[x]$.
- In $\operatorname{Spec} \mathbb{Z}$, each (p) is a closed point, while the generic point (0) is dense: every nonempty open set contains it.

The topology is therefore very different from the Euclidean topology. Its purpose is not to measure distance. It organizes algebraic specialization and vanishing.

Prime ideals and irreducible closed subsets

A nonempty closed subset Y of a topological space is called irreducible if it cannot be written as the union of two proper closed subsets. This notion gives a precise version of the idea that Y consists of one geometric piece.

Proposition

If $\mathfrak{p} \in \operatorname{Spec} A$, then $V(\mathfrak{p})$ is irreducible.

Proof. Suppose $V(\mathfrak{p}) = Y \cup Z$ with Y and Z closed in $V(\mathfrak{p})$. Since $\mathfrak{p}$ itself is a point of $V(\mathfrak{p})$, it lies in Y or in Z; say $\mathfrak{p} \in Y$. Because Y is closed in $V(\mathfrak{p})$, we may write $Y = V(\mathfrak{p}) \cap V(I)$ for some ideal $I \subset A$. From $\mathfrak{p} \in Y$ we get $I \subseteq \mathfrak{p}$. Every point $\mathfrak{q} \in V(\mathfrak{p})$ contains $\mathfrak{p}$, hence also contains I, so $\mathfrak{q} \in V(I)$. Therefore $V(\mathfrak{p}) \subseteq Y$, and hence $Y = V(\mathfrak{p})$. Thus $V(\mathfrak{p})$ is irreducible. $\square$

More generally,

$$V(I) \text{ is irreducible} \quad \Longleftrightarrow \quad \sqrt{I} \text{ is a prime ideal.}$$

Thus prime ideals are exactly the algebraic objects that encode irreducible closed subsets. The radical appears because $V(I)$ remembers only $\sqrt{I}$.

Exercise. Prove the converse: if $V(I)$ is irreducible, then $\sqrt{I}$ is prime. Hint: if $ab \in \sqrt{I}$, use $V(I) \subset V(ab) = V(a) \cup V(b)$.

There is also a useful point-set interpretation. The closure of the single point $\{\mathfrak{p}\}$ is exactly $V(\mathfrak{p})$:

$$\operatorname{cl}(\{\mathfrak{p}\}) = V(\mathfrak{p}).$$

So the point $\mathfrak{p}$ is a generic point of the whole irreducible closed subset $V(\mathfrak{p})$. This formalizes what we saw earlier in $\operatorname{Spec} k[x]$ and in the two components of $V(xy)$.

6 Noetherianity

A topological space is called *Noetherian* if every descending chain of closed subsets stabilizes.

Proposition

If A is a Noetherian ring, then $\operatorname{Spec} A$ is a Noetherian topological space.

Proof. Let

$$V(I_1) \supset V(I_2) \supset V(I_3) \supset \cdots$$

be a descending chain of closed subsets. Inclusion of closed sets reverses inclusion of radical ideals, so

$$\sqrt{I_1} \subset \sqrt{I_2} \subset \sqrt{I_3} \subset \cdots .$$

Because A is Noetherian, this ascending chain of ideals stabilizes, and hence so does the original chain of closed subsets. □

By the Hilbert Basis Theorem, $k[x_1, \ldots, x_n]$ is Noetherian. Therefore affine n-space $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ is a Noetherian topological space.

A useful consequence is that every closed subset of a Noetherian topological space is a finite union of irreducible closed subsets. Thus, in our setting, closed algebraic sets break into only finitely many irreducible components. We will use the statement, but not prove it here.

It is important that the converse of the proposition is false. The topology of $\operatorname{Spec} A$ only sees radical ideals. More precisely:

$$A \text{ is Noetherian} \iff \text{every ascending chain of ideals stabilizes,}$$

$$\operatorname{Spec} A \text{ is topologically Noetherian} \iff \text{every ascending chain of radical ideals stabilizes.}$$

Equivalently, $\operatorname{Spec} A$ is topologically Noetherian if and only if every radical ideal is the radical of a finitely generated ideal. Thus algebraic Noetherianity is strictly stronger in general.

Counterexample. Let

$$A = k[x_1, x_2, \ldots]/(x_i x_j : i, j \geq 1).$$

Write $\mathfrak{m} = (x_1, x_2, \ldots)$ in A. Since $\mathfrak{m}^2 = 0$, every prime ideal contains $\mathfrak{m}$; and since $A/\mathfrak{m} \cong k$, the ideal $\mathfrak{m}$ is maximal. Hence $\operatorname{Spec} A$ consists of a single point, so it is certainly a Noetherian topological space. However,

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$$

is a strictly ascending chain of ideals, so A is not a Noetherian ring.

The comparison also explains when the two conditions coincide: if every ideal of A is radical, then "ACC on ideals" and "ACC on radical ideals" are the same condition. For the affine schemes arising from finitely generated k-algebras, however, we will usually obtain algebraic Noetherianity directly from the Hilbert Basis Theorem.

7 Closed subspaces and quotient rings

There is an immediate geometric meaning to quotient rings. The quotient map

$$\pi : A \longrightarrow A/I$$

induces, by pullback of prime ideals, a map in the opposite direction

$$\pi^* : \operatorname{Spec}(A/I) \longrightarrow \operatorname{Spec} A.$$

Prime ideals of A/I are in one-to-one correspondence with prime ideals of A containing I:

$$\mathfrak{q}/I \longleftrightarrow \mathfrak{q}, \qquad I \subseteq \mathfrak{q}.$$

Hence the image of π^* is exactly

$$V(I) = \{\mathfrak{q} \in \operatorname{Spec} A : I \subseteq \mathfrak{q}\}.$$

In fact, π^* identifies $\operatorname{Spec}(A/I)$ homeomorphically with the closed subset $V(I)$.

A basic algebra–geometry dictionary

Passing from A to the quotient A/I corresponds geometrically to passing from $\operatorname{Spec} A$ to the closed subspace $V(I)$:

$$\boxed{\text{quotient ring } A/I} \qquad \longleftrightarrow \qquad \boxed{\text{closed subspace } V(I) \subseteq \operatorname{Spec} A}.$$

Later, after introducing structure sheaves, we will see this as a closed immersion of affine schemes.

8 Why a topological space is still not enough

There is one more problem. Compare the two rings

$$A = k[x]/(x) = k \qquad \text{and} \qquad B = k[x]/(x^2).$$

Both spectra have the same underlying point set: a single point. In fact, their underlying topological spaces are indistinguishable. But the rings are not the same. In B, the class of x is nonzero and nilpotent:

$$x \neq 0, \qquad x^2 = 0.$$

So the equation $x^2 = 0$ contains information that is lost if we remember only its ordinary solutions, and even if we remember only the topological space $\operatorname{Spec} B$.

Third principle

A geometric space must remember not only its points and closed sets, but also the functions defined on its open regions.

This is the point at which the structure sheaf will enter. We will not introduce it as an abstract gadget first. Instead, will ask a concrete question: what should the ring of functions on an open region of $\operatorname{Spec} A$ be? Localization will provide the first answer.

The next question

Yesterday we constructed the topological space $\operatorname{Spec} A$. Today we ask:

$$\boxed{\text{What are the functions on an open part of } \operatorname{Spec} A?}$$

The answer begins with localization and leads naturally to the structure sheaf.

9 Local functions and local rings

Let $\mathfrak{p} \in \operatorname{Spec} A$. To study functions near $\mathfrak{p}$, we should allow every denominator which does not vanish at $\mathfrak{p}$. Algebraically, this means inverting the multiplicative set

$$A \setminus \mathfrak{p}.$$

Definition

The **local ring of A at $\mathfrak{p}$** is

$$A_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1} A.$$

Its elements are fractions

$$\frac{a}{s}, \qquad s \notin \mathfrak{p}.$$

The ring $A_{\mathfrak{p}}$ has a unique maximal ideal,

$$\mathfrak{p}A_{\mathfrak{p}},$$

so it is genuinely a local ring. Its residue field is

$$\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong \operatorname{Frac}(A/\mathfrak{p}).$$

Interpretation

$A_{\mathfrak{p}}$ remembers only what can be detected in an arbitrarily small neighborhood of $\mathfrak{p}$. Elements outside $\mathfrak{p}$ are nonvanishing at $\mathfrak{p}$, so they become invertible.

Examples of local rings

For the closed point $(x-a)$ of $\mathbb{A}^1_k$,

$$k[x]_{(x-a)} = \left\{ \frac{g(x)}{h(x)} : g, h \in k[x],\ h(a) \neq 0 \right\}.$$

A rational expression $g(x)/h(x)$ is regular near a precisely when $h(a) \neq 0$ after cancellation.

At the generic point (0) of $\operatorname{Spec} \mathbb{Z}$,

$$\mathbb{Z}_{(0)} = \mathbb{Q}.$$

Thus the rational functions that appeared yesterday are literally the functions visible near the generic point.

At the closed point (p),

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : p \nmid b \right\}.$$

The only denominators allowed are those which do not vanish modulo p.

10 Principal open sets: localization becomes geometry

Let A be a commutative ring and let $f \in A$. The simplest open subsets of $\operatorname{Spec} A$ are the complements of the closed sets $V(f)$.

Definition

The **principal open subset** determined by f is

$$D(f) := \operatorname{Spec} A \setminus V(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}.$$

The basic identities for $V(-)$ immediately give

$$D(1) = \operatorname{Spec} A, \qquad D(0) = \varnothing, \qquad D(fg) = D(f) \cap D(g).$$

The last identity already looks like localization: to work on $D(f)$ we want to allow f to become invertible.

The fundamental correspondence

Consider the localization map

$$A \longrightarrow A_f = A\left[\frac{1}{f}\right].$$

By pullback of prime ideals it induces

$$\operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A.$$

Prime ideals of a localization correspond exactly to prime ideals of the original ring which are disjoint from the multiplicative set being inverted. Here the multiplicative set is

$$\{1, f, f^2, \ldots\},$$

so disjointness means precisely $f \notin \mathfrak{p}$.

Localization is restriction to a principal open

The natural map $A \to A_f$ identifies $\operatorname{Spec} A_f$ homeomorphically with $D(f) \subseteq \operatorname{Spec} A$:

$$\operatorname{Spec} A_f \cong D(f).$$

This is our first exact dictionary between an algebraic operation and an open geometric operation:

$$\boxed{A \longrightarrow A_f} \qquad \longleftrightarrow \qquad \boxed{\operatorname{Spec} A \supseteq D(f)}.$$

Regular functions on $D(f)$ are the elements of A_f. An element of A_f has the form

$$\frac{a}{f^n}.$$

This is the algebraic analogue of a rational expression whose denominator is allowed because f does not vanish anywhere on $D(f)$.

Three examples

1. The affine line. Let $A = k[x]$. Then

$$D(x) \cong \operatorname{Spec} k[x, x^{-1}].$$

If k is algebraically closed, the closed point (x) has been removed. Thus $D(x)$ is the affine line with the origin deleted.

2. Arithmetic geometry. In $\operatorname{Spec}\mathbb{Z}$,

$$D(6) \cong \operatorname{Spec}\mathbb{Z}\left[\frac{1}{6}\right].$$

The closed points (2) and (3) disappear, because both 2 and 3 become units. The generic point (0) remains.

3. The crossing $xy = 0$. Let

$$A = k[x, y]/(xy).$$

On $D(x)$ the element x is invertible. Since $xy = 0$, this forces $y = 0$, and hence

$$A_x \cong k[x, x^{-1}].$$

Geometrically, $D(x)$ sees the x-axis away from the origin; the other component has disappeared.

Principal opens form a basis

Principal opens are not merely convenient examples. They are enough to describe every open set locally.

Proposition

The sets $D(f)$, for $f \in A$, form a basis for the Zariski topology on $\operatorname{Spec} A$.

Proof. Let $U \subseteq \operatorname{Spec} A$ be open and let $\mathfrak{p} \in U$. Write the complement as

$$\operatorname{Spec} A \setminus U = V(I).$$

Since $\mathfrak{p} \notin V(I)$, the ideal I is not contained in $\mathfrak{p}$. Choose

$$f \in I \setminus \mathfrak{p}.$$

Since $f \notin \mathfrak{p}$, we have $\mathfrak{p} \in D(f)$. But $V(I) \subseteq V(f)$, so

$$\mathfrak{p} \in D(f) \subseteq U.$$

Thus every point of every open set has a principal-open neighborhood. □

Geometric meaning

The principal opens are the basic "coordinate neighborhoods" of an affine spectrum, and localization gives their rings of functions.

11 Sheaves

So far we know what functions should be on every principal open:

$$D(f) \quad \rightsquigarrow \quad A_f.$$

But an arbitrary open set $U \subseteq \operatorname{Spec} A$ need not itself be a single principal open. It is covered by many principal opens, and a function on U should be something that is regular on each sufficiently small piece and whose local descriptions agree where the pieces overlap.

This is exactly the situation that the notion of a sheaf is designed to organize.

First: what data do we need?

Suppose X is a topological space. For every open subset $U \subseteq X$, we want a ring $\mathcal{O}_X(U)$ of functions on U. Whenever

$$V \subseteq U,$$

there should be a restriction map

$$\rho^U_V : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V), \qquad s \longmapsto s|_V.$$

These restriction maps should behave as actual restrictions do:

$$\rho^U_U = \mathrm{id}, \qquad \rho^V_W \circ \rho^U_V = \rho^U_W \quad \text{for } W \subseteq V \subseteq U.$$

This is the data of a **presheaf of rings**. What turns it into a sheaf is one additional principle: functions are determined locally and can be glued from compatible local pieces.

The two sheaf axioms

Let $U = \bigcup_i U_i$ be an open cover.

Sheaf principle

A presheaf $\mathcal{O}_X$ is a **sheaf** if it satisfies the following two conditions.
Locality. If $s, t \in \mathcal{O}_X(U)$ satisfy

$$s|_{U_i} = t|_{U_i} \qquad \text{for every } i,$$

then $s = t$.
Gluing. If sections $s_i \in \mathcal{O}_X(U_i)$ satisfy

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \qquad \text{for every } i, j,$$

then there exists a unique $s \in \mathcal{O}_X(U)$ such that

$$s|_{U_i} = s_i \qquad \text{for every } i.$$

These axioms are not special to algebraic geometry. They formalize an elementary fact about ordinary functions: compatible local functions should determine one global function.

Why we need a sheaf here

A single ring A describes global functions on an affine space, but geometry also needs functions on smaller open regions. The structure sheaf stores all of these rings simultaneously, together with the information of how functions restrict and glue.

12 The structure sheaf on Spec A

Let

$$X = \operatorname{Spec} A.$$

The **structure sheaf** $\mathcal{O}_X$ is the sheaf of rings whose sections are the regular functions on open subsets of X.

There are two complementary ways to understand it.

The efficient description: principal opens

Because the principal opens form a basis, it is enough to know the structure sheaf on them. The fundamental formula is

$$\boxed{\mathcal{O}_X(D(f)) \cong A_f.}$$

In particular,

$$\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X) = \mathcal{O}_X(D(1)) \cong A.$$

Thus the original ring A is recovered as the ring of global regular functions on its affine spectrum.

Now suppose

$$D(g) \subseteq D(f).$$

Then

$$V(f) \subseteq V(g).$$

Hence $g^N \in (f)$ for some N, say

$$g^N = af.$$

Since g is invertible in A_g, we obtain

$$f^{-1} = \frac{a}{g^N} \in A_g.$$

Then f becomes invertible in A_g, so the universal property of localization gives a natural map

$$A_f \longrightarrow A_g.$$

Concretely, a fraction in A_f is simply regarded as the same fraction in A_g:

$$\frac{a}{f^n} \longmapsto \frac{a}{f^n}.$$

This is precisely the restriction map

$$\mathcal{O}_X(D(f)) \longrightarrow \mathcal{O}_X(D(g)).$$

Regular functions on an arbitrary open set

Now let $U \subseteq X$ be any open subset. Since the principal opens form a basis, we can cover U by principal opens:

$$U = \bigcup_{i \in I} D(f_i).$$

On each $D(f_i)$, a regular function is simply an element

$$s_i \in A_{f_i}.$$

These local functions define a regular function on all of U exactly when they agree on overlaps.

Since

$$D(f_i) \cap D(f_j) = D(f_i f_j),$$

both s_i and s_j can be restricted to

$$A_{f_i f_j}.$$

The compatibility condition is

$$s_i|_{D(f_i f_j)} = s_j|_{D(f_i f_j)} \qquad \text{for all } i, j.$$

Thus

$$\mathcal{O}_X(U) = \left\{ (s_i)_{i\in I} \in \prod_{i\in I} A_{f_i} : s_i = s_j \text{ in } A_{f_i f_j} \text{ for all } i, j \right\}.$$

In words: a regular function on U is obtained by giving regular functions on principal open pieces and requiring that they agree wherever the pieces overlap.

The sheaf property says that such compatible local functions glue to a unique regular function on U.

The basic picture

The structure sheaf is built from localization:

$$D(f) \longleftrightarrow A_f.$$

Restriction to a smaller open set corresponds to further localization, and functions on a general open set are obtained by gluing compatible functions on principal open subsets.

13 Stalks: the functions visible at one point

A sheaf contains a ring $\mathcal{O}_X(U)$ for every open set U. If we care only about what happens infinitesimally near a point $\mathfrak{p}$, we identify two sections whenever they agree on some sufficiently small neighborhood of $\mathfrak{p}$.

Definition

The **stalk** of $\mathcal{O}_X$ at $\mathfrak{p}$ is the ring of germs of regular functions near $\mathfrak{p}$:

$$\mathcal{O}_{X,\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in U} \mathcal{O}_X(U).$$

where the direct limit is over all open sets U containing $\mathfrak{p}$ via the restriction maps.

Two sections represent the same germ if they agree on some neighborhood of $\mathfrak{p}$.

For an affine spectrum, this abstract-looking construction is exactly the localization we already know.

Stalks are localizations

If $X = \operatorname{Spec} A$ and $\mathfrak{p} \in X$, then

$$\boxed{\mathcal{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}.}$$

Idea of the identification. If $s \notin \mathfrak{p}$, then

$$\mathfrak{p} \in D(s),$$

so a fraction $a/s \in A_{\mathfrak{p}}$ defines a regular function on the neighborhood $D(s)$ and hence a germ at $\mathfrak{p}$. Conversely, every germ is locally represented by such a fraction. Thus localization is precisely the algebra of germs at a point.

The local picture

There are now three natural scales:

$$A \longleftrightarrow \text{global functions on } \operatorname{Spec} A,$$

$$A_f \longleftrightarrow \text{functions on the basic open } D(f),$$
$$A_{\mathfrak{p}} \longleftrightarrow \text{germs of functions near the point } \mathfrak{p}.$$

14 Affine schemes

A **ringed space** is a topological space X together with a sheaf of rings $\mathcal{O}_X$. It is called a **locally ringed space** if every stalk $\mathcal{O}_{X,x}$ is a local ring.

For $X = \operatorname{Spec} A$, the stalk at $\mathfrak{p}$ is $A_{\mathfrak{p}}$, which is local. Hence the pair

$$(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

is a locally ringed space.

Definition

An **affine scheme** is a locally ringed space of the form

$$(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

for some commutative ring A.

This definition is much less mysterious after the route we have taken. The topological space came from prime ideals; localization told us what functions should look like on basic open sets; the sheaf organized those local functions and their gluing.

In particular, if $X = \operatorname{Spec} A$, every principal open is itself affine. With the restricted structure sheaf,

$$(D(f), \mathcal{O}_X|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f}).$$

Thus the same localization statement that began the day is now upgraded from a homeomorphism of topological spaces to an isomorphism of affine schemes.

Why the sheaf is genuinely necessary

Consider the rings:

$$A = k[x]/(x) \cong k, \qquad B = k[x]/(x^2) \cong k[\varepsilon]/(\varepsilon^2).$$

Their spectra have the same underlying topological space: one point. But their structure sheaves have different rings of global sections,

$$\Gamma(\operatorname{Spec} A, \mathcal{O}) = k, \qquad \Gamma(\operatorname{Spec} B, \mathcal{O}) = k[\varepsilon]/(\varepsilon^2).$$

At the unique point their stalks are also different. Thus the structure sheaf retains the nilpotent information that the topology alone forgets.

The point of schemes

A scheme is not just a set of solutions, and not just a topological space of prime ideals. It is a space together with a coherent system of local rings of functions.

Revisiting quotients

Recall that the quotient map

$$A \longrightarrow A/I$$

gives a homeomorphism

$$\operatorname{Spec}(A/I) \cong V(I) \subseteq \operatorname{Spec} A$$

onto a closed subset. With the structure sheaves included, this map is the basic example of a **closed immersion** of schemes. Thus the dictionary

$$\boxed{A/I} \qquad \longleftrightarrow \qquad \boxed{V(I)}$$

is not only topological: the quotient also records the functions on the closed subspace.

What is Next?

Affine schemes are only the local building blocks. Next we will learn how to glue affine pieces together, construct $\mathbb{P}^1$ from two copies of $\mathbb{A}^1$, and see a first genuinely non-affine geometric object.

15 An example

Before leaving affine schemes, let us revisit one example and collect the different pieces of the theory in one place.

Consider

$$A = k[x, y]/(xy), \qquad X = \operatorname{Spec} A.$$

Geometrically, this is the union of the two coordinate axes.

Points and irreducible components

The ideals

$$(x) \qquad \text{and} \qquad (y)$$

are prime ideals of A. They are the generic points of the two irreducible components

$$V(x) \qquad \text{and} \qquad V(y).$$

The maximal ideal

$$\mathfrak{m} = (x, y)$$

is the point where the two components meet.

Thus we have the specializations

$$(x) \subset (x, y), \qquad (y) \subset (x, y).$$

Equivalently,

$$\mathfrak{m} \in V(x) \cap V(y).$$

The equation $xy = 0$ is reflected topologically by

$$X = V(xy) = V(x) \cup V(y).$$

Open subsets and localization

Now consider the principal open subset $D(x)$. Since x is invertible there, the relation

$$xy = 0$$

forces

$$y = 0.$$

Hence

$$A_x = \left(k[x, y]/(xy)\right)_x \cong k[x, x^{-1}],$$

and therefore

$$D(x) \cong \operatorname{Spec} k[x, x^{-1}].$$

Geometrically, $D(x)$ is the x-axis with the origin removed. The entire y-axis disappears.

Similarly,

$$D(y) \cong \operatorname{Spec} k[y, y^{-1}],$$

which is the y-axis with the origin removed.

Notice that

$$D(x) \cap D(y) = D(xy) = \varnothing,$$

since $xy = 0$ in A.

The local ring at the intersection point

At the point

$$\mathfrak{m} = (x, y),$$

the local ring is

$$\mathcal{O}_{X,\mathfrak{m}} \cong A_{\mathfrak{m}} = \left(k[x, y]/(xy)\right)_{(x,y)}.$$

This ring remembers both branches meeting at the origin. Neither x nor y becomes invertible at $\mathfrak{m}$, and the relation

$$xy = 0$$

remains visible in the local ring.

What this example collects

For the affine scheme

$$X = \operatorname{Spec} k[x, y]/(xy),$$

prime ideals describe irreducible pieces, localization describes open regions, stalks describe neighborhoods of individual points, and the structure sheaf keeps all of these local rings organized in one geometric object.

16 The projective line

Gluing affine schemes

Not every geometric object should be expected to be affine.

The basic idea is the same one used throughout geometry: construct a larger space by taking simpler pieces and identifying parts of them.

Suppose U and V are affine schemes. Assume that we have open subsets

$$U' \subseteq U, \qquad V' \subseteq V,$$

together with an isomorphism

$$\varphi : U' \xrightarrow{\sim} V'.$$

We may form a new space by identifying each point of U' with its image under φ.

But we must glue more than the underlying topological spaces. The regular functions must also be identified through the same isomorphism. In this way the structure sheaves on U and V combine to give a sheaf on the glued space.

Gluing principle

Affine schemes are local building blocks.
If two affine schemes contain isomorphic open pieces, we may identify those pieces and glue both the spaces and their regular functions.

For two pieces, this is the entire idea. In more complicated gluings one must require compatibility on triple overlaps, but we will not need that here.

Our main example is obtained by gluing two copies of the affine line.

Constructing the projective line

Take two copies of the affine line:

$$U_0 = \operatorname{Spec} k[t], \qquad U_1 = \operatorname{Spec} k[s].$$

Inside them consider the principal opens

$$D(t) \subseteq U_0, \qquad D(s) \subseteq U_1.$$

By localization,

$$D(t) \cong \operatorname{Spec} k[t, t^{-1}]$$

and

$$D(s) \cong \operatorname{Spec} k[s, s^{-1}].$$

These open subsets are isomorphic. On coordinate rings, the isomorphism is given by

$$k[s, s^{-1}] \longrightarrow k[t, t^{-1}], \qquad s \longmapsto t^{-1}.$$

Geometrically, this means that on the overlap we identify the two coordinates by

$$s = \frac{1}{t}.$$

We now glue U_0 and U_1 along these open subsets.

Definition

The resulting space is the **projective line** over k, denoted

$$\mathbb{P}^1_k.$$

Thus

$$\mathbb{P}^1_k = U_0 \cup U_1,$$

where

$$U_0 \cong \mathbb{A}^1_k, \qquad U_1 \cong \mathbb{A}^1_k,$$

and their intersection is

$$U_0 \cap U_1 \cong \operatorname{Spec} k[t, t^{-1}].$$

The point $(s) \in U_1$ does not belong to $D(s)$, so it is not identified with any point of U_0. It is the new point added to the affine line, usually denoted by

$$\infty.$$

Thus one may picture

$$\mathbb{P}^1_k = \mathbb{A}^1_k \cup \{\infty\}.$$

The second affine coordinate

$$s = \frac{1}{t}$$

is precisely the coordinate that remains well behaved near the point at infinity.

Global regular functions on $\mathbb{P}^1_k$

The gluing description allows us to compute the global regular functions very easily.

A global section

$$F \in \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k})$$

is determined by a regular function on each affine chart:

$$f(t) \in k[t], \qquad g(s) \in k[s].$$

These two functions must agree on the overlap. Under the identification

$$s = t^{-1},$$

this means

$$f(t) = g(t^{-1})$$

inside the Laurent polynomial ring

$$k[t, t^{-1}].$$

But

$$f(t) \in k[t]$$

contains only nonnegative powers of t, whereas

$$g(t^{-1}) \in k[t^{-1}]$$

contains only nonpositive powers of t.

Therefore the equality can hold only when both functions are constant. Equivalently,

$$k[t] \cap k[t^{-1}] = k$$

inside $k[t, t^{-1}]$.

We have proved:

Global functions on the projective line

Every global regular function on $\mathbb{P}^1_k$ is constant:

$$\boxed{\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = k.}$$

$\mathbb{P}^1_k$ is not affine

This immediately gives our first example of a scheme which is not affine.

Proposition

The projective line $\mathbb{P}^1_k$ is not an affine scheme.

Proof. Recall that if

$$X = \operatorname{Spec} A$$

is affine, then

$$\Gamma(X, \mathcal{O}_X) \cong A.$$

Thus an affine scheme is recovered from its ring of global functions:

$$X \cong \operatorname{Spec} \Gamma(X, \mathcal{O}_X).$$

If $\mathbb{P}^1_k$ were affine, then the previous theorem would give

$$\mathbb{P}^1_k \cong \operatorname{Spec} \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = \operatorname{Spec} k.$$

But $\operatorname{Spec} k$ consists of a single point, whereas $\mathbb{P}^1_k$ contains the open subset

$$U_0 \cong \mathbb{A}^1_k,$$

which certainly has more than one point. This is impossible. Hence $\mathbb{P}^1_k$ is not affine. □

First genuinely non-affine example

Each of the two charts U_0 and U_1 is affine, but their union $\mathbb{P}^1_k$ is not.
So being affine is not a local property: affine pieces can glue together to produce a non-affine geometric object.

17 Schemes

We are now ready for the general definition.

Definition

A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ has an open neighborhood U for which

$$(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

for some commutative ring A.

In other words, a scheme is a space which is *locally affine*.

This is analogous to the way a manifold is locally modeled on Euclidean space. Here the local models are not open subsets of $\mathbb{R}^n$, but affine schemes.

Why $\mathbb{P}^1_k$ is a scheme

By construction,

$$\mathbb{P}^1_k = U_0 \cup U_1,$$

where

$$U_0 \cong \operatorname{Spec} k[t] \cong \mathbb{A}^1_k$$

and

$$U_1 \cong \operatorname{Spec} k[s] \cong \mathbb{A}^1_k.$$

Thus every point of $\mathbb{P}^1_k$ lies in an open subset which is an affine scheme. Therefore $\mathbb{P}^1_k$ is a scheme. At the same time, we have just proved that it is not itself affine.

The local-to-global viewpoint

Affine schemes are the local coordinate pieces of algebraic geometry.
General schemes are obtained by allowing these affine pieces to be glued together.