A Construction of the Category of Mixed Tate Motives Using Aomoto Polylogarithms and Nori Formalism

by

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A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Doctor of Philosophy

in

Mathematics

KOÇ ÜNİVERSİTESİ

August 18, 2023

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ABSTRACT

A Construction of the Category of Mixed Tate Motives Using Aomoto Polylogarithms and Nori Formalism
Ahmet Berkay Kebeci
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Grothendieck proposed the category of motives as a Tannakian category, offering a universal framework for cohomology theories in the realm of algebraic geometry. In this study, we will consider motives in the sense of Nori. One expects the Hopf algebra of mixed Tate motives to be isomorphic to the bi-algebra $A_\bullet$ of Aomoto polylogarithms. We aim to construct $A_\bullet$ using Nori motives and Nori's diagram formalism.

ÖZETÇE

Birleşik Tate Motifleri Kategorisinin, Aomoto Polilogaritmaları ve Nori Formalizmi Kullanılarak İnşası

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Matematik, Doktora
18 Ağustos 2023

Grothendieck, cebirsel geometri alanındaki kohomoloji teorileri için evrensel bir çerçeve sunan Tannakacı bir kategori olarak motifler kategorisini önerdi. Bu çalışmamızda motifleri Nori'nin tanımıyla ele alacağız. Birleşik Tate motifleri kategorisinin Hopf cebirinin, $A_\bullet$ ile gösterilen Aomoto polilogaritmaları cebirine izomorf olması bekleniyor. $A_\bullet$ cebirini Nori motiflerini ve Nori'nin diagram formalizmini kullanarak inşa etmeyi amaçlıyoruz.

ACKNOWLEDGMENTS

I wish to wholeheartedly express my gratitude to my advisor, *Sinan Ünver*, for embarking on enlightening mathematical journeys and for his unwavering motivation and patience throughout this research expedition. His exceptional guidance and profound expertise have been indispensable at every juncture of my research and the creation of this thesis.

I extend my sincere appreciation to *Burak Özbağcı*, *Tolga Etgü*, *İlhan İkeda*, and *Ayberk Zeytin* for generously agreeing to be part of my thesis committee. I am also grateful to *Ergün Yalçın* for his influential guidance during my undergraduate education.

My deepest gratitude goes to my parents, *Birgül Kebeci* and *Faik Kebeci*, my brother *Ayberk Kebeci*, my grandparents *Ahmet Yerli* and *Feride Yerli*, and my uncle *Özgür Yerli* for their unwavering love, continuous support, and ready availability whenever needed.

Special acknowledgment is reserved for *Nihan Tanısalı*, whose love has not only been a pillar of strength but has also played an instrumental role in my personal growth.

I am indebted to my friends from Koç University - *İhsan*, *Umut*, *Fırtına*, *Ruhat*, *Duygu*, *Muratcan*, *Murad*, *Alperen*, *Mutlu*, *Aslı*, *Begüm*, *Keremcan*, *Cem*, *Şeyma*, *Rızacan*; my friends from Bilkent University - *Fatih*, *Selçuk*, *Ali*, *Celal Selçuk*, *Halil*, *Ahmet*, *Edip*; my high school friends *Semih*, *Onuralp*; my neighbors *Elif*, *Alper*, *Duru*, *Seval*, *Veysel* and my childhood friend *Zeynep* for their unwavering support and the wonderful times we have shared. I also want to express gratitude to the cats I and my friends have shared our homes with - *Arte*, *Sirius*, *Düğme*, *Patik*, *Tontuş*, *Pars*, *Köfte*, *Dumduma*, *Geveze*, *Gema*, *Tilki* - for providing us solace and relief from daily stress.

Lastly, I thank Technological Research Council of Turkey (TÜBİTAK) for their financial support.

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Chapter 0

INTRODUCTION

In 1964, Grothendieck introduced the concept of a "motif" in a letter to Serre, and he later described them as the most enigmatic and perhaps the most powerful entities of discovery among the objects he had the privilege of uncovering. It is originally proposed to address Weil conjectures. The theory of motives is expected to serve as a universal cohomology theory for varieties and play a fundamental role akin to Galois theory for periods, employing Tannakian formalism.

Although the existence of such a category has yet to be established, several candidates, including Nori motives, Deligne motives, Voevodsky motives, and Ayoub motives, among others, have been proposed. Within this context, our focus will be on Nori motives, with particular attention to the special case of mixed Tate motives, the full subcategory of motives consisting of mixed Tate objects.

In this thesis, we explain Nori's construction of motives using quiver representations. Furthermore, we will explore Aomoto polylogarithms, which constitute the anticipated Hopf algebra of the category of mixed Tate motives. Finally, we construct this algebra via limits of (cohomological) Nori motives, with the expectation that it will be realized using Nori's formalism on quiver representations. Consequently, we propose a construction for mixed Tate motives.

Periods

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals

$$\int_\sigma f(x_1, ..., x_n) dx_1 ... dx_n,$$

where f is a rational function with rational coefficients and $\sigma \subseteq \mathbb{R}^n$ is given by polynomial inequalities with rational coefficients. For example,

$$\begin{aligned}\sqrt{2} &= \int_{2x^2 \leq 1} dx, \\ \pi &= \int_{x^2+y^2 \leq 1} dxdy, \\ \log(2) &= \int_1^2 \frac{dx}{x}, \\ \zeta(2) &= \int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2}, \\ \zeta(2,1) &= \int_{1 \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3}.\end{aligned}$$

are periods. Periods form a subring of $\mathbb{C}$ and we will denote the ring of periods by $\mathcal{P}^{\text{eff}}$. Since the set of rational functions with rational coefficients is countable, so is $\mathcal{P}^{\text{eff}}$. Most of the numbers used in number theory are periods. First of all $\overline{\mathbb{Q}} \subset \mathcal{P}^{\text{eff}}$. Some of the basic examples of families that are periods are logarithms of rational numbers, and zeta values, more generally, multiple zeta values. For a more comprehensive and detailed introduction to periods, we refer to [KZ01].

Another important family of periods is polylogarithms of rational numbers. Polylogarithms is a generalization of logarithm defined inductively by

$$\ell i_1(z) = -\log(1-z)$$

and

$$d\ell i_n(z) = \ell i_{n-1}(z)\frac{dz}{z},$$

with $\ell i_n(0) = 0$. It turns out that polylogarithms play a critical role in the theory of mixed Tate motives.

An equivalent way of defining periods, using cohomology, is the following. For a smooth $\mathbb{Q}$-variety X and a normal crossing divisor $Y \subseteq X$, we have a canonical isomorphism

$$H^i_{\text{dR}}(X,Y) \otimes_{\mathbb{Q}} \mathbb{C} \to H^i_{\text{B}}(X,Y;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

known as the period isomorphism. By the universal coefficient theorem, there is a map

$$H_i^{\text{B}}(X,Y;\mathbb{Q}) \otimes H^i_{\text{B}}(X,Y;\mathbb{Q}) \to \mathbb{Q}$$

and by period isomorphism, this induces

$$\begin{aligned} H^i_{\mathrm{dR}}(X,Y) \otimes H^{\mathrm{B}}_i(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Q}) &\to \mathbb{C} \\ \omega \otimes \sigma &\mapsto \int_\sigma \omega. \end{aligned}$$

This is called the *period pairing.* We call a *period* of (X,Y) any element in the image of the period pairing.

Let us consider the pair $(X,Z) = (\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,\infty\}, \{1,q\})$, with $q \in \mathbb{Q} \setminus \{0,1\}$. First de Rham cohomology of $(X,Z) = (\operatorname{Spec}\mathbb{Q}[x,x^{-1}], \{1,q\})$ has a basis $\{\omega_1,\omega_2\}$, where $\omega_1 = \frac{dt}{t}, \omega_2 = \frac{dt}{q-1}$. First singular homology of $(X(\mathbb{C}),Z(\mathbb{C})) = (\mathbb{C}^*, \{1,q\})$ has a basis $\{\sigma_1,\sigma_2\}$, where σ_1 is a (counterclockwise) circle around 0 with radius $r < \min\{1,|q|\}$ and σ_2 is the straight line from 1 to q. Hence this pair gives the matrix

$$\begin{pmatrix} \int_{\sigma_2}\omega_2 & \int_{\sigma_2}\omega_1 \\ \int_{\sigma_1}\omega_2 & \int_{\sigma_1}\omega_1 \end{pmatrix} = \begin{pmatrix} 1 & \log q \\ 0 & 2\pi\mathrm{i} \end{pmatrix}$$

which shows that logarithms of rational numbers and $2\pi\mathrm{i}$ are periods.

If a complex number is a period, there are many ways to write it as an integral. However, checking whether two periods are equal is not easy. For example $\pi\sqrt{163}$ and $3 \cdot \log(640320)$ both have decimal expensions beginning 40.10916999113251... but they are not equal. ($e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007...$ is known as the Ramanujan constant, which is very close to an integer.) A famous conjecture, known as the period conjecture, states that if a period has two integral representations, one can pass between them using only the following calculus rules.

– Additivity of integral:

$$\int_\sigma \omega_1 + \omega_2 = \int_\sigma \omega_1 + \int_\sigma \omega_2$$
$$\int_{\sigma_1 \cup \sigma_2} \omega = \int_{\sigma_1} \omega + \int_{\sigma_2} \omega$$

where $\sigma_1 \cap \sigma_2 = \emptyset$.

– Change of variables:

$$\int_{f(\sigma)} \omega = \int_\sigma f^*\omega$$

where f is invertible and defined by polynomial equations with rational coefficients.

– Stokes' formula:

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

An equivalent formulation of this is the following, due to Kontsevich-Zagier. The ring of *abstract effective periods* $\mathcal{P}_{\mathrm{KZ}}^{\mathrm{eff}}$ is the $\mathbb{Q}$-vector space generated by symbols $[X, Y, i, \sigma, \omega]$, where X is a $\mathbb{Q}$-variety, $Y \subseteq X$ a closed subvariety, $i \in \mathbb{N}$, $\sigma \in H_i^{\mathrm{B}}(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Q})$ and $\omega \in H^i_{\mathrm{dR}}(X, Y)$, subject to the following relations:

– (Additivity) The map $(\sigma, \omega) \mapsto [X, Y, i, \sigma, \omega]$ is bilinear.

– (Change of variables) For any morphism $f : X \to X'$ such that $f(Y) \subseteq Y'$,

$$[X, Y, i, \sigma, f^*\omega'] = [X', Y', i, f_*\sigma, \omega'].$$

– (Stokes' formula) For any chain $X \supseteq Y \supseteq Z$ of closed subvarieties,

$$[Y, Z, i, \sigma, \delta\omega] = [X, Y, i+1, \partial\sigma, \omega],$$

where δ and ∂ are connecting morphisms.

The multiplication is defined by

$$[X, Y, i, \sigma, \omega][X', Y', i', \sigma', \omega'] = [X \times X', X \times Y' \cup X' \times Y, i + i', \sigma \times \sigma', \omega \wedge \omega'].$$

The ring of *abstract periods* is the localization $\mathcal{P}_{\mathrm{KZ}}$ of $\mathcal{P}_{\mathrm{KZ}}^{\mathrm{eff}}$ with respect to

$$[\mathbb{G}_m, \{1\}, 1, S^1, \frac{dx}{x}].$$

The $\mathbb{Q}$-algebra morphism

$$\begin{aligned} \mathrm{ev} : \mathcal{P}_{\mathrm{KZ}} &\to \mathbb{C} \\ [X, Y, i, \sigma, \omega] &\mapsto \int_\sigma \omega \end{aligned}$$

is called the *evaluation map*. The image of this map is $\mathcal{P} := \mathcal{P}^{\mathrm{eff}}[\frac{1}{2\pi \mathrm{i}}]$. The period conjecture holds if and only if the evaluation map is injective.

Motives

The theory of periods is related to the (partly conjectural) theory of motives. The concept of 'motive' is introduced by Grothendieck as the 'universal cohomology theory for varieties'. The category of mixed motives $\mathrm{MM}(k)$ over a field k is a conjectural Tannakian category, together with a contravariant functor $h : \mathrm{Var}_k \to \mathrm{MM}(k)$ such that any Weil cohomology theory factors through h. A candidate for the theory of motives is Nori motives. It is a Tannakian category and any cohomology theory that can be compared with singular cohomology (such as de Rham cohomology) factors through it.

Singular cohomology and de Rham cohomology induce functors

$$f_B, f_{dR} : \mathrm{MM}(\mathbb{Q}) \to \mathbb{Q} - \mathrm{Mod}\,.$$

(Note that any cohomology theory having a comparison isomorphism with singular cohomology factors through the category of mixed Nori motives. Hence, considering motives in Nori's sense, the functors f_B and f_{dR} already exist.) Then for any motive $M \in \mathrm{MM}(\mathbb{Q})$, the period pairing yields a pairing

$$f_B(M)^\vee \otimes f_{dR}(M) \to \mathbb{C}.$$

Let $\mathcal{P}(M)$ be the subfield of $\mathbb{C}$ generated by the image of the pairing. The period conjecture is equivalent to Grothendieck's period conjecture, that is the following. $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain and for any (Nori) motive M,

$$\mathrm{trdeg}[\mathcal{P}(M) : \mathbb{Q}] = \dim G_{\mathrm{mot}}(M),$$

where $G_{\mathrm{mot}}(M) = \mathrm{Aut}^{\otimes} H_{B|\langle M\rangle}$ is the Galois group of the Tannakian subcategory $\langle M\rangle$ of $\mathrm{MM}(\mathbb{Q})$ generated by M. See [Ayo14] for more details. This implies a 'Galois theory for periods' and suggests that algebraic relations between periods come from motives. See [And09] for more details of this theory.

The category of Nori motives is defined as follows. Let k be a subfield of $\mathbb{C}$. Let X be a k-variety, $Y \subseteq X$ be a closed subvariety and i be an integer. We call (X, Y, i) an *effective pair*. Let $\mathrm{Pairs}^{\mathrm{eff}}$ be the directed graph whose vertices are effective pairs and whose edges are the following. For any morphism $f : X \to X'$ such that $f(Y) \subseteq Y'$, we have an edge $(X', Y', i) \to (X, Y, i)$. For any chain $X \supseteq Y \supseteq Z$ of closed subvarieties, an edge $(Y, Z, i) \to (X, Y, i+1)$. The relative singular cohomology

$$\begin{aligned} H^* : \mathrm{Pairs}^{\mathrm{eff}} &\to \mathbb{Z} - \mathrm{Mod} \\ (X, Y, i) &\mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}) \end{aligned}$$

is a quiver representation (i.e., it is a functor from the path category of $\mathrm{Pairs}^{\mathrm{eff}}$) to the category of finitely generated $\mathbb{Z}$-modules. The category of *effective mixed Nori motives* $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(k)$ (or just $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$) is defined as the diagram category $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$, i.e., H^* factorises over

$$\mathrm{Pairs}^{\mathrm{eff}} \xrightarrow{\tilde{H}} \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} \xrightarrow{f_H} \mathbb{Z} - \mathrm{Mod}$$

where f_H is faithful and exact and $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is the universal Abelian category with this property. Explicitly, this is the $2-\mathrm{colim}$ of the categories $\mathrm{End}(H^*|_F) - \mathrm{Mod}$ where F runs over the finite full subgraphs of $\mathrm{Pairs}^{\mathrm{eff}}$ and f_H is the forgetful functor. We call $H^i_{\mathrm{Nori}}(X, Y) := \tilde{H}(X, Y, i)$.

We define a tensor product on $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ in the light of the Künneth formula. We call $\mathbf{1}(-1) := H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{1\}) \in \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$. Finally, the category $\mathcal{MM}_{\mathrm{Nori}}(k)$ of *mixed Nori motives* is defined as the localization of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ with respect to $\mathbf{1}(-1)$.

Mixed Tate motives and Aomoto polylogarithms

There is a weight filtration $W_\bullet$ on the mixed Nori motives with rational coefficients, compatible with the weight filtration on their Hodge realizations. We say that M is a *mixed Tate (Nori) motive* if $\mathrm{gr}^W_{2n} M$ is a direct sum of copies of $\mathbf{1}(-n) := \mathbf{1}(-1)^{\otimes n}$. We denote the full subcategory of such objects by $\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}$. This category is also Tannakian. Furthermore, it is a mixed Tate category (see section 1.1). By the Tannakian formalism of mixed Tate categories, the Galois group of $\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}$ is $U \rtimes \mathbb{G}_m$, where $U = \mathrm{Spec}\, R$ such that $\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}$ is equivalent to the category of graded R-comodules.

We call an n-*simplex* a family of $n+1$ hyperplanes in $\mathbb{P}^n$. Let (L, M) be a pair of n-simplices not having a common face. Then

$$H^n_{\text{Nori}}\left(\mathbb{P}^n \setminus L, M \setminus (L \cap M)\right)$$

is a mixed Tate motive. gr^W_0 of this motive gives the period

$$a(L, M) = \int_{\Delta_M} \omega_L$$

where Δ_M is the n-simplex defined by M and $\omega_L = d\log(z_1/z_0) \wedge ... \wedge d\log(z_n/z_0)$ given that $z_i = 0$ is a homogeneous equation of L_i. For instance, letting $q \in \mathbb{Q}\backslash\{0, 1\}$, take $M_q = \bigcup M_i$ and $L = \bigcup L_i$, where $L_j : z_i = 0$, $M_0 : z_0 = z_1; M_1 : z_0 = z_1 + z_2; M_i : z_i = z_{i+1}$ for $2 \leq i < n$; and $M_n : qz_0 = z_n$. Then $a(L, M) = \ell i_n(q)$.

The algebra of Aomoto polylogarithms is defined by mimicking the behaviors of the integrals $a(L, M)$. Its n-th grade A_n is the abelian group generated by symbols $(L; M)$ for such pairs of simplices as above such that the following relations hold.

– If the hyperplanes of one of L or M are not in general position (i.e. degenerate), then $(L; M) = 0$.

– For every $\sigma \in S_n$,

$$(\sigma L; M) = (L; \sigma M) = (-1)^{|\sigma|}(L; M)$$

where σL and σM, are defined by the natural action of S_n on a set indexed by $1, ..., n$.

– For every family of hyperplanes $L_0, ..., L_{n+1}$ and an n-simplex M,

$$\sum (-1)^j (\hat{L}^j; M) = 0,$$

where $\hat{L}^j = (L_0, ..., \hat{L}_j, ..., L_{n+1})$, and the corresponding relation for the second component.

– For every $g \in \text{PGL}_{n+1}(k)$,

$$(gL; gM) = (L; M).$$

Let $A = \oplus A_n$. There is a multiplication defined so that it corresponds to the multiplication of corresponding periods. Also, comultiplication is defined as compatible with the Hodge realizations of the corresponding motives. These make A a Hopf algebra. There is expected to be an isomorphism of Hopf algebras $R \to A_{\mathbb{Q}}$.

A candidate for a construction of mixed Tate motives using Nori formalism

Following the idea in Madhav Nori's letter [Nor04] to Sinan Ünver, we aim to construct 'the category of mixed Tate motives'. For this, it suffices to construct its Hopf algebra $R = \bigoplus_{d \geq 0} R_d$. We consider the mixed Tate motives arising from the following configurations. Fix $n \in \mathbb{N}^{>0}$. Let $B = \bigcup_{1 \leq i \leq m} B_i$, where all B_i are hyperplanes in B that meet $x_{i_1} = ... = x_{i_k} = 0$ properly for all $\{i_1, ..., i_k\} \subseteq \{1, ..., n\}$. Let

$$M = \bigoplus_{d \geq 0} M_d$$

where

$$M_d = \mathrm{gr}^W_{2n-2d}(\varprojlim_B H^n_{\mathrm{Nori}}\,(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m)) \otimes \mathbf{1}(n-d)$$

such that the limit is taken over all such B. Thus

$$M_0 = \mathbf{1}(0)$$

and

$$M_n = \mathrm{gr}^W_0(\varprojlim_B H^n_{\mathrm{Nori}}\,(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m)).$$

We can view M as a R-comodule and since $M_0 = \mathbf{1}(0)$, there is a map $M_n \to R_n$. The group $G_n := S_n \ltimes \mathbb{G}^n_m$ acts on $\mathbb{G}^n_m$ and therefore on M_n. Let

$$R'_n := H_0(G_n; M_n).$$

be the 0-th group homology. The map $M_n \to R_n$ induces a map

$$\varphi_n : R'_n \to R_n.$$

Let $R' = \bigoplus_{n \geq 0} R'_n$ and $\varphi = \bigoplus \varphi_n$. We show that R' and A are isomorphic as graded algebras. Therefore, R' has a structure of a Hopf algebra. We conjecture that

$$\varphi \otimes \mathbb{Q} : R' \otimes \mathbb{Q} \to R$$

is an isomorphism of graded Hopf algebras.

Chapter-by-chapter summary

Chapter 1 gives some category theoretical tools that will be useful in the latter chapters. In section 1.1, we explain mixed Tate categories and their Tannakian formalism. The main reference for this section is chapter 3 of [Gon01]. In section 1.2 and 1.3, we examine the diagram formalism of Nori. Everything in these sections can be found in [HMS17]. Another essential reference for these sections is [Nor00].

Chapter 2 investigates the Nori motives. First, we define Nori motives. Then we show that we can do the same construction using specific pairs called good pairs (and very good pairs). Then we present some properties such as being Tannakian and universal. We finish the chapter by giving the weight filtration and introducing mixed Tate motives. We mostly follow [HMS17] and [Nor00].

Chapter 3 starts with classical polylogarithms. In section 3.2, we will briefly introduce Aomoto polylogarithms. The primary references of this section are [BGSV07] and [Gon95]. Section 3.3 is the main part of the thesis. We present our construction following [Nor04].

Notation.

- For an abelian group A, we denote $A_{\mathbb{Q}} := A \otimes_{\mathbb{Z}} \mathbb{Q}$.

- For a ring R, let $R_m := R[t]/(t^m)$.

- For an abelian category $\mathcal{A}$ and a functor $F : R - \text{Alg} \to \mathcal{A}$, where $R - \text{Alg}$ is the category of R-algebras, we define

$$F(R_m)^{\circ} := \ker(F(R_m) \to F(R))$$

as the *infinitesimal part* of $F(R_m)$.

- Let k be a field. We denote the category of k-varieties by Var_k (or just Var). We denote the category of affine k-varieties by Aff_k (or just Aff).

- Let $\mathcal{A}$ be an abelian category. We denote the category of bounded chain complexes in $\mathcal{A}$ by $C^b(\mathcal{A})$, the bounded homotopy category of $\mathcal{A}$ by $K^b(\mathcal{A})$ and the bounded derived category of $\mathcal{A}$ by $D^b(\mathcal{A})$.

Chapter 1

CATEGORY THEORETICAL PRELIMINARIES

1.1 Mixed Tate Categories

Let k be a field of characteristic 0 and $\mathcal{M}$ be a Tannakian k-category with an invertible object $k(1)$. Set $k(n) := k(1)^{\otimes n}$, for any $n \in \mathbb{Z}$. We call $(\mathcal{M}, k(1))$ a *mixed Tate category* if

- $k(n)$ are mutually non-isomorphic
- any simple object of $\mathcal{M}$ is isomorphic to one of $k(n)$,
- $\mathrm{Hom}_{\mathcal{M}}(k(n), k(n)) = k$, for any $n \in \mathbb{Z}$,
- $\mathrm{Ext}^1_{\mathcal{M}}(k(0), k(n)) = 0$ if $n < 0$.

Note that since $\mathcal{M}$ is a rigid tensor category, it is equivalent to its opposite category by $X \mapsto X^\vee$. Therefore, for $M \in \mathcal{M}$, the functor $- \otimes M$ is exact. Hence the last condition is equivalent with

- $\mathrm{Ext}^1_{\mathcal{M}}(k(m), k(n)) = 0$ if $n < m$.

Let $M \in \mathcal{M}$. Since $\mathcal{M}$ is abelian, we can find a filtration of M such that its grades are simple. Using the condition on Ext^1, we can reorganize this and find a filtration $W_\bullet M$, indexed by $2\mathbb{Z}$, such that each grade $\mathrm{gr}^W_{2n} M = W_{2n}M/W_{2n-2}M$ is a direct sum of copies of $k(-n)$. We call this the *weight filtration* of M. Note that morphisms are compatible with this filtration.

A *pure functor* between mixed Tate categories is a tensor functor $\varphi : (M_1, k(1)_1) \to (M_2, k(1)_2)$ with an isomorphism $\varphi(k(1)_1) \to k(1)_2$.

1.1.1 Graded vector spaces

One of the simplest examples of a mixed Tate category is the category of graded vector spaces over k, which we denote by $k\text{-Vect}_\bullet$. The objects of this category are k-vector spaces $V = \bigoplus_n V_n$, where each V_n is also a k-vector space. The morphisms are k-linear maps $f : V \to W$ such that $f(V_i) \subseteq W_i$ for all i.

The Tannakian Galois group of $k\text{-Vect}_\bullet$ is $\mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$. To show this, by Tannaka duality, it suffices to show that $k\text{-Vect}_\bullet \simeq \operatorname{Rep} \mathbb{G}_m (\simeq \text{Comod-}k[x, x^{-1}])$. For a graded vector space V, define the map $\rho : V \to V \otimes k[x, x^1]$, given by $\rho(a) = a \otimes x^n$ if $a \in V_n$. Then (V, ρ) is a $k[x, x^{-1}]$-comodule. On the other hand, let (V, ρ) be a $k[x, x^{-1}]$-comodule. Since $\mathbb{G}_m$ is diagonalizable, $V = \bigoplus_\chi V_\chi$, where χ runs through the characters of $\mathbb{G}_m$ and

$$V_\chi = \{v \in V : \rho(v) = v \otimes \mathfrak{a}(\chi)\}.$$

(See [Mil12, Theorem XIV.4.7].) Here, $\mathfrak{a}(\chi)$ denotes the image of x in the map $k[x, x^{-1}] \to k[x, x^{-1}]$ induced by χ. But $\mathfrak{a}(\chi) = x^n$ for some $n \in \mathbb{Z}$, so $V = \bigoplus_\chi V_\chi$ gives a grading.

1.1.2 Galois group of a mixed Tate category

Let $\mathcal{M}$ be a mixed Tate category. Consider the following pure functor.

$$\begin{aligned} \psi : \mathcal{M} &\to k\text{-Vect}_\bullet \\ M &\mapsto \bigoplus_n \operatorname{Hom}_{\mathcal{M}}(k(-n), \operatorname{gr}^W_{2n} M) \end{aligned}$$

Let $\tilde{\psi} : \mathcal{M} \to k\text{-Vect}$ be its composition with the forgetful functor. Then $\tilde{\psi}$ is a fiber functor.

Theorem 1.1.1. $\operatorname{Aut}^\otimes(\tilde{\psi}) \simeq U(\mathcal{M}) \rtimes \mathbb{G}_m$, *where* $U(\mathcal{M})$ *is a pro-unipotent group scheme.*

Proof. Consider the embedding

$$\begin{aligned} \iota : k\text{-Vect} &\hookrightarrow \mathcal{M} \\ V_n &\mapsto V_n \otimes k(-n). \end{aligned}$$

We have

$$\begin{aligned}(\psi\circ\iota)(V_n) &= \psi(V_n\otimes k(-n))\\ &= \mathrm{Hom}_{\mathcal{M}}(k(-n), V_n\otimes k(-n))\\ &\simeq \mathrm{Hom}_{\mathcal{M}}(k(-n), k^{\dim V_n}\otimes k(-n))\\ &\simeq \mathrm{Hom}_{\mathcal{M}}(k(-n), k(-n)^{\dim V_n})\\ &\simeq \mathrm{Hom}_{\mathcal{M}}(k(-n), k(-n))^{\dim V_n}\\ &\simeq k^{\dim V_n}\\ &\simeq V_n.\end{aligned}$$

This gives a splitting $\mathbb{G}_m \hookrightarrow \mathrm{Aut}^{\otimes}(\tilde{\psi}) \twoheadrightarrow \mathbb{G}_m$ due to [Mil12, Proposition X.4.1] and [Mil12, Proposition X.4.3]. Call $U(\mathcal{M}) := \ker(\mathrm{Aut}^{\otimes}(\tilde{\psi}) \twoheadrightarrow \mathbb{G}_m)$. Let $M \in \mathcal{M}$ and $\langle M\rangle$ be the Tannakian subcategory of $\mathcal{M}$ generated by M. Since $U(\mathcal{M})$ acts trivially on the simple objects $k(n)$, the group scheme $U(\mathcal{M})\mid_{\langle M\rangle}$ is unipotent. Thus $U(\mathcal{M})$ is pro-unipotent. □

According to the Tannakian formalism, $\mathcal{M}$ is equivalent to the category of finite dimensional modules over $\mathrm{Aut}^{\otimes}(\tilde{\psi})$ and it is equivalent to the category of graded finite dimensional modules over $U(\mathcal{M})$.

$\mathcal{U}_{\bullet}(\mathcal{M}) := \mathrm{End}(\psi)$ is a graded Hopf algebra. The graded dual Hopf algebra $\mathcal{U}_{\bullet}(\mathcal{M})^{\vee} := \oplus_{k\geq 0}\mathcal{U}_k(\mathcal{M})^{\vee}$ is isomorphic to the Hopf algebra of the pro-group scheme $U(\mathcal{M})$.

1.1.3 Framed objects

Let n be a non-negative integer and $M \in \mathcal{M}$. We call M an *n-framed* object if there are non-zero maps $v_o : k(0) \to \mathrm{gr}_0^W M$ and $f_n : \mathrm{gr}_{-2n}^W M \to k(n)$.

Let M_1 and M_2 be n-framed objects. Define the coarsest equivalence relation satisfying $M_1 \sim M_2$ if there is a morphism $M_1 \to M_2$ respecting the frames. Let $\mathcal{A}_n(\mathcal{M})$ be the set of equivalence classes of n-framed objects in $\mathcal{M}$.

Let $n > 0$ and $[M, v_0, f_n]$ be an n-framed object. Since $k(0), k(n)$ are simple and v_0, f_n are non-zero, v_0 is an injection and f_n is a surjection. Let N_1 be the intermediate object in $W_{-2}M \subseteq W_0M$ corresponding $\operatorname{Im} v_0$ and N_2 be the intermediate object in $W_{-2n-2}M \subseteq W_{-2n}M$ corresponding $\ker f_n$. Then $N_1 \hookrightarrow M$ and $N_1 \twoheadrightarrow N_1/N_2$ respect frames. So $M \sim N_1/N_2$. Therefore any n-framed object is equvalent to one M with $W_0M = M$, $W_{-2n-2} = 0$, $\operatorname{gr}_0^W M = k(0)$ and $\operatorname{gr}_{-2n}^W M = k(n)$.

$\mathcal{A}_n(\mathcal{M})$ is an ableian group with the following addition.

$$[M, v_o, f_n] + [M', v_0', f_n,] = [M \oplus M', (v_0, v_0'), f_n + f_n'].$$

Then, the unit is $[k(0) \oplus k(n), \mathrm{id}_{k(0)}, \mathrm{id}_{k(n)}]$ and the inversion is given by the following formula.

Proposition 1.1.2. $-[M, v_0, f_n] = [M, -v_0, f_n] = [M, v_0, -f_n]$.

Proof. We can assume that $W_0M = M$, $W_{-2n-2} = 0$, $\operatorname{gr}_0^W M = k(0)$ and $\operatorname{gr}_{-2n}^W M = k(n)$. We will show that $[M \oplus M, (v_0, v_0), f_n - f_n]$ is equivalent to $[k(0) \oplus k(n), \mathrm{id}_{k(0)}, \mathrm{id}_{k(n)}]$. Let D be the diagonal of $M \oplus M$. Then, the quotient map $M \oplus M \to (M \oplus M)/W_{-2}D =: N_1$ respects the frames. Let $W_{-2}N_1 \subset N \subset N_1$ be such that $N/W_{-2}N_1$ is the diagonal of $\operatorname{gr}_0^W N_1 = N_1/W_{-2}N_1 = k(0) \oplus k(0)$. Then, the inclusion map $N \to N_1$ respects the frames.

Because of the condition on Ext^1, there is a map $\alpha : k(0) \hookrightarrow N$ such that the composition $k(0) \xrightarrow{\alpha} N \twoheadrightarrow \operatorname{gr}_0^W N = k(0)$ is the identity map. Let $\beta : k(n) = W_{-2n}N \hookrightarrow N$ be the natural inclusion map. The map $(\alpha, \beta) : k(0) \oplus k(n) \to N$ respects the frames. □

If M is a 0-framed object, we may assume that $\operatorname{gr}_0^W M = M$. Moreover, the mapping

$$\mathcal{A}_0(\mathcal{M}) \to \operatorname{End}(k(0)) = k$$
$$[M, v_0, f_0] \mapsto f_0 \circ v_0$$

is an isomorphism.

$\mathcal{A}_\bullet(\mathcal{M})$ is a graded Hopf algebra. The tensor product induces commutative multiplication. Comultiplication is defined as

$$\Delta = \bigoplus_{0 \le k \le n} \Delta_{k,n-k} : \mathcal{A}_n(\mathcal{M}) \to \bigoplus_{0 \le k \le n} \mathcal{A}_k(\mathcal{M}) \otimes \mathcal{A}_{n-k}(\mathcal{M})$$

in the following way. Choose a basis $\{b_i\}_{1\leq i\leq m}$ for $\mathrm{Hom}_{\mathcal{M}}(k(p), \mathrm{gr}^W_{-2p} M)$ and the dual basis $\{b'_i\}_{1\leq i\leq m}$ for $\mathrm{Hom}_{\mathcal{M}}(\mathrm{gr}^W_{-2p} M, k(p))$. Define

$$\begin{aligned}\Delta_{p,n-p} : \mathcal{A}_n(\mathcal{M}) &\to \mathcal{A}_p(\mathcal{M}) \otimes \mathcal{A}_{n-p}(\mathcal{M}) \\ [M, v_o, f_n] &\mapsto \sum_{i=1}^{m} [M, v_0, b'_i] \otimes [M, b_i, f_n](-p).\end{aligned}$$

Here $[M, b_i, f_n](-p) := [M(-p), b_i(-p), f_n(-p)]$, where $M(-p) := M \otimes k(-p)$,

$$b_i(-p) : k(0) = k(p) \otimes k(-p) \xrightarrow{b_i \otimes \mathrm{id}} (\mathrm{gr}^W_{-2p} M) \otimes k(-p) = \mathrm{gr}^W_0 M(-p)$$

and

$$f_n(-p) : gr^W_{-2n+2p} M(-p) = (gr^W_{-2n} M) \otimes k(-p) \xrightarrow{f_n \otimes \mathrm{id}} k(n) \otimes k(-p) = k(n-p).$$

The unit is $1 \in k = \mathcal{A}_0$ and the counit is the projection map $\mathcal{A} \to \mathcal{A}_0 = k$.

Theorem 1.1.3. *The map*

$$\varphi : \mathcal{A}_\bullet(\mathcal{M}) \to \mathcal{U}_\bullet(\mathcal{M})^\vee$$

given by

$$\varphi([M, v_0, f_n])(F) = f_n \circ F_m(v_0) \in \mathrm{End}(k(n)) = k$$

is an isomorphism of Hopf algebras.

See [Gon01, Theorem 3.2] for the proof.

1.1.4 Mixed Tate objects

Let $\mathcal{C}$ be any Tannakian k-category and $k(1)$ be a rank 1 object of $\mathcal{C}$ such that $k(i) := k(1)^{\otimes i}$ are mutually non-isomorphic. An object M of $\mathcal{C}$ is called a *mixed Tate object* if it admits a finite increasing filtration $W_\bullet$ indexed by $2\mathbb{Z}$ such that gr^W_{2n} is a direct sum of copies of $k(-n)$. Denote by $T\mathcal{C}$ the full subcategory of mixed Tate objects of $\mathcal{C}$. Then, $(T\mathcal{C}, k(1))$ is a mixed Tate category.

Example 1.1.4. We will construct the category of Hodge-Tate structures using the recipe above. Take $\mathcal{C} = \mathrm{MHS}_{\mathbb{Q}}$, the category of mixed Hodge structures over $\mathbb{Q}$.

A *Hodge-Tate structure* over $\mathbb{Q}$ is a mixed Tate object in $\mathrm{MHS}_{\mathbb{Q}}$. Equivalently, a mixed Hodge structure H is a Hodge-Tate structure if and only if $h^{p,q} = 0$ when $p \neq q$, where $h^{p,q} := \dim \mathrm{gr}_F^p \mathrm{gr}_{p+q}^W H_{\mathbb{C}}$ are the Hodge numbers of H. We denote the category of Hodge-Tate structures over $\mathbb{Q}$ by $\mathrm{HT}_{\mathbb{Q}}$. Thus $(\mathrm{HT}_{\mathbb{Q}}, \mathbb{Q}(1))$ is a mixed Tate category. See the first section of [BGSV07], for the *Hodge-Tate algebra* $\mathcal{A}_{\bullet}(\mathrm{HT}_{\mathbb{Q}})$.

1.2 Nori Diagrams

We assume all rings to be commutative and Noetherian, and all categories to be small.

Definition 1.2.1.

1. A *diagram* is a directed graph (or quiver).

2. A *subdiagram* is a subgraph.

3. A subdiagram is called *full* if it contains all edges between its vertices.

4. A diagram is called *finite* if it has only finitely many vertices.

5. A *representation* T of a diagram D in a category $\mathcal{C}$ is a function $T : D \to \mathcal{C}$ such that it induces a functor from the path category $\mathcal{P}(D) \to \mathcal{C}$. (This is also known as quiver representation.)

Theorem 1.2.2. *Let D be a diagram, R be a ring and $T : D \to R - \mathrm{Mod}$ be a representation. Then, there is an R-linear abelian category $\mathcal{C}(D,T)$ with representation $\tilde{T} : D \to \mathcal{C}(D,T)$ and a faithful, exact, R-linear functor $f_T : \mathcal{C}(D,T) \to R - \mathrm{Mod}$ such that T factorises as*

$$T : D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R - \mathrm{Mod}$$

and $\mathcal{C}(D,T)$ is universal with this property, i.e., for any

1. *R-linear abelian category $\mathcal{A}$,*

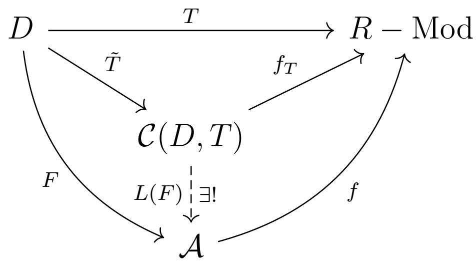
2. faithful, exact, R-linear functor $f : \mathcal{A} \to R - \mathrm{Mod}$*,*

3. representation $F : D \to \mathcal{A}$

satisfying the factorization

$$T : D \xrightarrow{F} \mathcal{A} \xrightarrow{f} R - \mathrm{Mod}$$

there is a faithful, exact functor $L(F)$*-unique up to unique isomorphism of exact functors- such that*



commutes.

Moreover, $\mathcal{C}(D,T)$ *together with* $\tilde{T}$ *and* f_T *is unique up to unique equivalence of categories.*

Before proving this, we will give the explicit description of $\mathcal{C}(D,T)$. Fix a diagram D, a ring R and a representation $T : D \to R - \mathrm{Mod}$. We define

$$\mathrm{End}(T) := \{(e_p)_{p\in D} \in \prod_{p\in D} \mathrm{End}_R(Tp) \mid \begin{array}{ccc} Tp & \xrightarrow{Tm} & Tq \\ \downarrow{\scriptstyle e_p} & & \downarrow{\scriptstyle e_q} \\ Tp & \xrightarrow{Tm} & Tq \end{array} \text{ commutes, } \forall p,q \in D, m \in D(p,q)\}.$$

In other words, $\mathrm{End}(T)$ is the kernel of the map

$$\phi : \prod_{p\in D} \mathrm{End}_R(Tp) \to \prod_{p,q\in D} \prod_{m\in D(p,q)} \mathrm{Hom}_R(Tp, Tq)$$

given by

$$\phi(p)(m) = e_q \circ T_m - T_m \circ e_p.$$

Note that $\mathrm{End}(T)$ is an R-algebra. Since each Tp is finitely generated and R is Noetherian, each $\mathrm{End}_R(Tp)$ is also finitely generated. Therefore, if D is finite, $\mathrm{End}(T)$ is finitely generated as an R-module. Now, we can describe $\mathcal{C}(D,T)$.

Definition 1.2.3.

1. If D is finite,
$$\mathcal{C}(D,T) := \mathrm{End}(T) - \mathrm{Mod}\,.$$

2. In general,
$$\mathcal{C}(D,T) := 2 - \mathrm{colim}_F\, \mathcal{C}(F, T\big|_F),$$
where F runs through finite full subdiagrams of D, i.e., the objects of $\mathcal{C}(D,T)$ are the objects of $\mathcal{C}(F, T\big|_F)$ for some F and the morphisms are
$$\mathrm{Mor}_{\mathcal{C}(D,T)}(X,Y) = \varinjlim_F \mathrm{Mor}_{\mathcal{C}(F,T|_F)}(X_F, Y_F),$$
where X_F is the image of $X \in \mathcal{C}(F', T\big|_{F'})$ in $\mathcal{C}(F, T\big|_F)$ for $F \supseteq F'$.

$\mathcal{C}(D,T)$ is called the *diagram category.*

By definition, we have $\mathcal{C}(D,T) = \mathcal{C}(\mathcal{P}(D), T)$.

Each $p \in D$ has $\mathrm{End}(T\big|_F)$-action on it, for any subdiagram $F \subseteq D$ with $p \in F$. So,

$$\begin{aligned} \tilde{T} : D &\to \mathcal{C}(D,T) \\ p &\mapsto Tp \end{aligned}$$

defines a representation and this gives a factorization

$$T : D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R - \mathrm{Mod}$$

of T, where f_T is the forgetful functor.

Proposition 1.2.4. *The diagram category $\mathcal{C}(D,T)$ agrees with its smallest full abelian subcategory $\mathcal{C}$ containing the image of $\tilde{T}$ such that $f_T\big|_{\mathcal{C}}$ is exact.*

See [HMS17, Proposition 7.3.24] for the proof. Therefore, each object of $\mathcal{C}(D,T)$ is a subquotient of a finite direct sum of objects from $\{\tilde{T}p \mid p \in D\}$.

In the case that $\mathcal{P}(D)$ is R-linear abelian and T is faithful, exact, R-linear, the categories $\mathcal{P}(D)$ and $\mathcal{C}(D,T)$ are equivalent by the following theorem. See [Nor00, Proposition 1.10] or [HMS17, Theorem 7.1.20] for the proof.

Theorem 1.2.5. *Let $\mathcal{A}$ be an R-linear abelian category and $T : \mathcal{A} \to R - \mathrm{Mod}$ be a faithful, exact, R-linear functor. If*

$$T : \mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A}, T) \xrightarrow{f_T} R - \mathrm{Mod}$$

is the factorization of T via its diagram category, then $\tilde{T}$ is an equivalence of categories.

We will show that $\mathcal{C}(D, T)$ in definition 1.2.3 satisfies the universal property in theorem 1.2.2.

Proof of theorem 1.2.2. First, we will construct $L(F)$.

Lemma 1.2.6. *Let D and D' be finite diagrams and $F : D' \to D$ be a diagram morphism. Let $T : D \to R - \mathrm{Mod}$ be a representation.*

$$\begin{array}{ccc} D' & \xrightarrow{F} & D \\ & \searrow^{T'} & \downarrow{\scriptstyle T} \\ & & R - \mathrm{Mod} \end{array}$$

Then

$$T' = T \circ F : D' \to R - \mathrm{Mod}$$

is a representation and F induces an R-algebra morphism

$$F^* : \mathrm{End}(T) \to \mathrm{End}(T').$$

Proof. F induces

$$\prod_{p \in D} \mathrm{End}_R(Tp) \to \prod_{p' \in D'} \mathrm{End}_R(T'p')$$
$$e = (e_p)_p \mapsto F^*(e)$$

where $(F^*(e))_{p'} = e_{F(p')}$. ∎

Lemma 1.2.7. *Let D_1 and D_2 be diagrams and $G : D_1 \to D_2$ be a diagram morphism. Let $T : D_2 \to R - \mathrm{Mod}$ be a representation.*

$$\begin{array}{ccc} D_1 & \xrightarrow{G} & D_2 \\ & \searrow^{T \circ G} & \downarrow{\scriptstyle T} \\ & & R - \mathrm{Mod} \end{array}$$

Then G induces a faithful, exact, R-linear functor $\mathcal{G} : \mathcal{C}(D_1, T \circ G) \to \mathcal{C}(D_2, T)$ such that

$$\begin{array}{ccc} D_1 & \xrightarrow{\quad G \quad} & D_2 \\ \downarrow{\scriptstyle \widetilde{T\circ G}} & & \downarrow{\scriptstyle \tilde{T}} \\ \mathcal{C}(D_1, T \circ G) & \overset{\mathcal{G}}{\dashrightarrow} & \mathcal{C}(D_2, T) \\ & {\scriptstyle f_{T\circ G}} \searrow \quad \swarrow {\scriptstyle f_T} & \\ & R - \mathrm{Mod} & \end{array}$$

commutes.

Proof. If D_1 and D_2 are finite, then by lemma 1.2.6, there is a map

$$G^* : \mathrm{End}(T) \to \mathrm{End}(T \circ G)$$

which induces a functor

$$\mathcal{C}(D_1, T \circ G) = \mathrm{End}(T \circ G) - \mathrm{Mod} \to \mathrm{End}(T) - \mathrm{Mod} = \mathcal{C}(D_2, T).$$

If D_1 is finite, D_2 is arbitrary, then let E_2 be a full subdiagram of D_2 with $G(D_1) \subseteq E_2 \subseteq D_2$. By finite case, $G : D_1 \to E_2$ induces a functor

$$\mathcal{C}(D_1, T \circ G) \to \mathcal{C}(E_2, T\big|_{E_2}).$$

Composing this with $\mathcal{C}(E_2, T\big|_{E_2}) \to \mathcal{C}(D_2, T)$, which comes from colimit, we get the desired functor.

If D_1 and D_2 are arbitrary, for any finite subdiagram $E_1 \subseteq D_1$, there is a functor

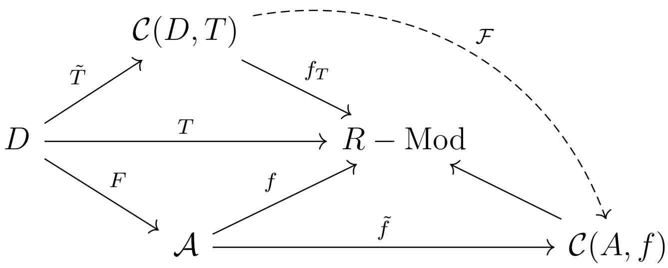
$$\mathcal{C}(E_1, (T \circ G)\big|_{E_1}) \to \mathcal{C}(D_2, T)$$

by the previous case. By colimit, this extends to a functor

$$\mathcal{C}(D_1, T \circ G) \to \mathcal{C}(D_2, T).$$

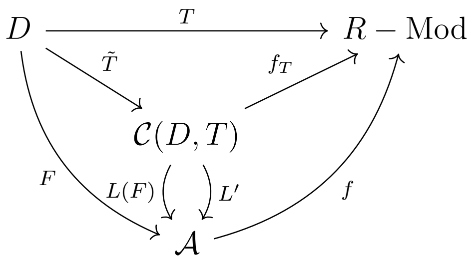
□

By lemma 1.2.7, F induces a functor $\mathcal{F} : \mathcal{C}(D, T) \to \mathcal{C}(\mathcal{A}, f)$ such that



commutes. By theorem 1.2.5, $\tilde{f}$ is an equivalence of categories. Finally, define $L(F)$ as the composition of $\mathcal{F}$ with the inverse of $\tilde{f}$.

Next, we will show that $L(F)$ is unique up to unique isomorphism of exact functors. Let L' be a functor satisfying the same conditions as $L(F)$. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}(D,T)$ such that $L(F)\big|_{\mathcal{C}'} = L'\big|_{\mathcal{C}'}$. We want to show that $\mathcal{C}' = \mathcal{C}(D,T)$.



Let $x, y \in \mathcal{C}'$ and $m : x \to y \in \mathcal{C}(D,T)$. Then $L(F)(x) = L'(x) =: x'$ and $L(F)(y) = L'(y) =: y'$. Also $(f \circ L(F))(m) = f_T(m) = (f \circ L')(m) : f(x') \to f(y')$. Since f is faithful, we have $L(F)(m) = L'(m)$. Hence $m \in \mathcal{C}'$. Thus $\mathcal{C}'$ is a full subcategory of $\mathcal{C}(D,T)$.

If $p \in D$, then $L(F)(\tilde{T}p) = F(p) = L'(\tilde{T}p)$, so $\tilde{T}p \in \mathcal{C}$. Since $L(F)$ and L' are exact, they also agree on the subquotients of finite direct sums of objects of the form $\tilde{T}p$, up to isomorphism. Hence, the result follows by proposition 1.2.4. $\square$

Let D_2 be a diagram and $T_2 : D_2 \to R - \mathrm{Mod}$ be a representation with the factorization

$$T_2 : D_2 \xrightarrow{\tilde{T}_2} \mathcal{C}(D_2, T_2) \xrightarrow{f_{T_2}} R - \mathrm{Mod}\,.$$

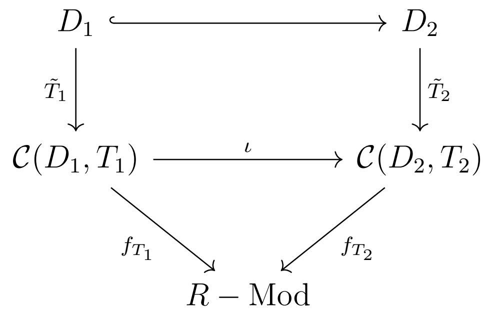
Let $D_1 \subseteq D_2$ be a a full subdiagram and $T_1 = T_2\big|_{D_1}$ with the factorization

$$T_1 : D_1 \xrightarrow{\tilde{T}_1} \mathcal{C}(D_1, T_1) \xrightarrow{f_{T_1}} R - \mathrm{Mod}\,.$$

By lemma 1.2.7, there is a faithful, exact, R-linear functor

$$\iota : \mathcal{C}(D_1, T_1) \to \mathcal{C}(D_2, T_2)$$

such that



commutes.

Corollary 1.2.8. *Let D_1, D_2, T_1, T_2 and ι be as above. If there is a representation*

$$F : D_2 \to \mathcal{C}(D_1, T_1)$$

with an isomorphism

$$T_2 \to f_{T_1} \circ F$$

of functors, then ι is an equivalence of categories.

See [HMS17, Corollary 7.1.19] for the proof.

1.2.1 The diagram category as a category of comodules

Let D be a diagram, R be a field or a Dedekind domain, and

$$T : D \to R - \mathrm{Proj}$$

be a representation. Then,

$$\begin{aligned} \mathcal{C}(D, T) :&= 2 - \mathrm{colim}_F(\mathrm{End}(T\big|_F) - \mathrm{Mod}) \\ &= 2 - \mathrm{colim}_F(\mathrm{End}(T\big|_F)^\vee - \mathrm{Comod}) \\ &= (\varinjlim_F (\mathrm{End}(T\big|_F)^\vee) - \mathrm{Comod} \end{aligned}$$

where F runs through finite full subdiagrams of D (see [HMS17, Corollary 7.5.7]). We call

$$A(D, T) := \varinjlim_F (\mathrm{End}(T\big|_F)^\vee).$$

1.3 Tensor Structure of the Diagram Category

In this section, we assume that any diagram D has a distinguished edge $\mathrm{id}_v : v \to v$, for each $v \in D$, called the *identity edge* of v. We also assume that a morphism of diagrams (i.e., a morphism of directed graphs) maps identity edge to identity edge.

1.3.1 Product structure

Definition 1.3.1. A *graded diagram* D is a diagram with a function

$$|\cdot| : D \to \mathbb{Z}/2\mathbb{Z}.$$

If $\lambda : u \to v$ is an edge from D, then $|\lambda| := |u| - |v|$.

Definition 1.3.2. Given diagrams D_1 and D_2, their *direct product* $D_1 \times D_2$ is the diagram with

- vertices (v_1, v_2), for each $v_1 \in D_1$, $v_2 \in D_2$,
- edges (α, id), for each edge α in D_1 and (id, β), for each edge β in D_2,
- $\mathrm{id} = (\mathrm{id}, \mathrm{id})$.

This is the product in the category of diagrams. There are natural maps.

$$\begin{array}{ccccc} v_1 & & \longleftarrow\!\!\!\shortmid & & (v_1, v_2) \\ D_1 & \longleftarrow & D_1 \times D_2 & & \downarrow \\ & & \downarrow & & \\ & & D_2 & & v_2 \end{array}$$

There is a natural grading on $D_1 \times D_2$ given by $|(v_1, v_2)| = |v_1| + |v_2|$.

Definition 1.3.3. Let D be a graded diagram. A *commutative product structure* on D is a diagram map

$$D \times D \to \mathcal{P}(D)$$

$$(v, w) \mapsto v \times w$$

with a vertex $1 \in D$ (called *unit*) and choices of edges

$$\alpha_{v,w} : v \times w \to w \times v$$

$$\beta_{v,w,u} : v \times (w \times u) \to (v \times w) \times u$$

$$\beta'_{v,w,u} : (v \times w) \times u \to v \times (w \times u)$$

$$u_v : v \to 1 \times v$$

for all $v, w, u \in D$.

Fix a ring R.

Definition 1.3.4. Let D be a graded diagram with a commutative product structure. A *graded multiplicative representation* T of D is a representation

$$T : D \to R - \mathrm{Proj}$$

with a choice of isomorphism

$$\tau_{v,w} : T(v \times w) \xrightarrow{\sim} T(v) \otimes T(w)$$

for each $v, w \in D$, such that:

1.

$$\begin{array}{ccc} T(v \times w) & \xrightarrow{T(\alpha_{v,w})} & T(w \times v) \\ \downarrow \wr & & \downarrow \wr \\ T(v) \otimes T(w) & \xrightarrow{(-1)^{|v||w|}\psi} & T(w) \otimes T(v) \end{array}$$

where $\psi : T(v) \otimes T(w) \xrightarrow{\sim} T(w) \otimes T(v)$ is the natural isomorphism of R-modules, commutes.

2. For any edge $\lambda : v \to v'$,

$$\begin{array}{ccc} T(v \times w) & \xrightarrow{T(\lambda \times \mathrm{id})} & T(v' \times w) \\ \downarrow \wr & & \downarrow \wr \\ T(v) \otimes T(w) & \xrightarrow{(-1)^{|\lambda||w|}T(\lambda)\otimes \mathrm{id}} & T(v') \otimes T(w) \end{array}$$

commutes.

3. For any edge $\lambda : v \to v'$,

$$\begin{array}{ccc} T(w \times v) & \xrightarrow{T(\mathrm{id} \times \lambda)} & T(w \times v') \\ \downarrow \sim & & \downarrow \sim \\ T(w) \otimes T(v) & \xrightarrow{\mathrm{id} \otimes T(\lambda)} & T(w) \otimes T(v') \end{array}$$

commutes.

4.

$$\begin{array}{ccc} T(v \times (w \times u)) & \xrightarrow{T(\beta_{v,w,u})} & T((v \times w) \times u) \\ \downarrow \sim & & \downarrow \sim \\ T(v) \otimes T(w \times u) & & T(v \times w) \otimes T(u) \\ \downarrow \sim & & \downarrow \sim \\ T(v) \otimes (T(w) \otimes T(u)) & \xrightarrow{\phi} & (T(v) \otimes T(w)) \otimes T(u) \end{array}$$

where ϕ is the natural isomorphism, commutes.

5. $T(\beta_{v,w,u}) = T(\beta'_{v,w,u})^{-1}$.

6. There is an isomorphism $R \xrightarrow{\sim} T(1)$ such that

$$\begin{array}{ccc} R \otimes T(v) & \xrightarrow{\sim} & T(1) \otimes T(v) \\ \downarrow \sim & & \downarrow \sim \\ T(v) & \xrightarrow{T(u_v)} & T(1 \times v) \end{array}$$

commutes.

Proposition 1.3.5. *Let D be a graded diagram with a commutative product structure and*

$$T : D \to R - \mathrm{Proj}$$

be a graded multiplicative representation. Then,

1. *$\mathcal{C}(D,T)$ is a tensor category with a tensor functor*

$$T : \mathcal{C}(D,T) \to R - \mathrm{Mod}\,.$$

2. *If R is a field or a Dedekind domain, then $A(D,T)$ is a bialgebra and the scheme $\mathrm{Spec}(A(D,T))$ is a faithfully flat unital monoid scheme over $\mathrm{Spec}\,R$.*

See [HMS17, Proposition 8.1.5] for the proof.

1.3.2 Localization

Let D^{eff} be a graded diagram with a commutative product structure and $v_0 \in D^{\mathrm{eff}}$. The *localized diagram* D with respect to v_0 is defined as the graded diagram with the following vertices and edges

- For each $v \in D^{\mathrm{eff}}$ and $n \in \mathbb{Z}$, a vertex $v(n) \in D$ with $|v(n)| = |v|$.
- For each $\alpha : v \to w$ in D^{eff} and $n \in \mathbb{Z}$, an edge $\alpha(n) : v(n) \to w(n)$ in D.
- For each $v \in D^{\mathrm{eff}}$ and $n \in \mathbb{Z}$, an edge $(v \times v_0)(n) \to v(n+1)$.

having the following commutative product structure

$$D \times D \to \mathcal{P}(D)$$
$$(v(n), w(m)) \mapsto (v \times w)(n+m)$$

with unit $1(0)$ and

$$\alpha_{v(n),w(m)} = \alpha_{v,w}(n+m)$$
$$\beta_{v(n),w(m),u(r)} = \beta_{v,w,u}(n+m+r)$$
$$\beta'_{v(n),w(m),u(r)} = \beta'_{v,w,u}(n+m+r)$$
$$u_{v(n)} = u_v(n).$$

There is a natural inclusion of diagrams

$$D^{\mathrm{eff}} \to D$$
$$v \mapsto v(0).$$

Proposition 1.3.6. *Let R be a field or a Dedekind domain and*

$$T : D^{\text{eff}} \to R - \text{Proj}$$

be a graded multiplicative representation such that $T(v_0)$ is locally free of rank 1. Then $\mathcal{C}(D,T)$ is the localization of $\mathcal{C}(D^{\text{eff}},T)$ with respect to $T(v_0)$, where

$$\begin{aligned} T : D &\to R - \text{Proj} \\ v(n) &\mapsto T(v) \otimes T(v_0)^{\otimes n} \\ \alpha(n) &\mapsto T(\alpha) \otimes T(\text{id})^{\otimes n}. \end{aligned}$$

See [HMS17, Proposition 8.2.5] for the proof.

Chapter 2

NORI MOTIVES

This chapter will give a definition and some proprieties of Nori motives. We assume all rings to be commutative and Noetherian, and all categories to be small. Fix a subfield k of $\mathbb{C}$, with an embedding $k \hookrightarrow \mathbb{C}$. By a *$k$-variety* we mean a reduced separated scheme of finite type over k.

2.1 Nori Motives

Let X be a k-variety, $Y \subseteq X$ be a closed subvariety and i be an integer. We call (X, Y, i) an *effective pair*. Let $\text{Pairs}^{\text{eff}}$ be the diagram whose vertices are effective pairs and edges are the following. For any morphism $X \to X'$ such that $f(Y) \subseteq Y'$, we have an edge $(X', Y', i) \to (X, Y, i)$. For any chain $X \supseteq Y \supseteq Z$ of closed subvarieties, an edge $(Y, Z, i) \to (X, Y, i+1)$.

The relative singular cohomology

$$\begin{aligned} H^* : \text{Pairs}^{\text{eff}} &\to \mathbb{Z} - \text{Mod} \\ (X, Y, i) &\mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}) \end{aligned}$$

is a representation. We define the category of *effective mixed Nori motives* as

$$\mathcal{MM}^{\text{eff}}_{\text{Nori}}(k) := \mathcal{C}(\text{Pairs}^{\text{eff}}, H^*).$$

For an effective pair (X, Y, i), we write $H^i_{\text{Nori}}(X, Y)$ for the corresponding object in $\mathcal{MM}^{\text{eff}}_{\text{Nori}}$. We denote $H^i_{\text{Nori}}(X) := H^i_{\text{Nori}}(X, \emptyset)$. The category $\mathcal{MM}_{\text{Nori}}(k)$ of *mixed Nori motives* is defined as the localization of $\mathcal{MM}^{\text{eff}}_{\text{Nori}}$ with respect to $\mathbf{1}(-1) := H^1_{\text{Nori}}(\mathbb{G}_m, \{1\}) \in \mathcal{MM}^{\text{eff}}_{\text{Nori}}$. For $M \in \mathcal{MM}_{\text{Nori}}$ and $r \in \mathbb{N}$, we denote $r \cdot M := M^{\oplus r}$.

2.2 Good and Very Good Pairs

Definition 2.2.1.

1. We call an effective pair (X, Y, i) an *effective good pair* if $H^j(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}) = 0$ for $j \neq i$ and $H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$ is free.

2. An effective good pair is called an *effective very good pair* if X is affine, $X \setminus Y$ is smooth and either $\dim(X) = i$, $\dim(Y) = i - 1$ or $X = Y$ and $\dim(X) < i$.

3. We denote the full subdiagram of $\text{Pairs}^{\text{eff}}$ with effective good pairs by Good^{eff}.

4. We denote the full subdiagram of Good^{eff} with effective very good pairs by $\text{VGood}^{\text{eff}}$.

Restricting H^* to Good^{eff} (or $\text{VGood}^{\text{eff}}$), it takes values in the category of free $\mathbb{Z}$-modules which is $\mathbb{Z} - \text{Proj}$.

2.2.1 Very good filtrations

We can look at the diagram categories of Good^{eff} and $\text{VGood}^{\text{eff}}$. It turns out the resulting categories $\mathcal{C}(\text{Pairs}^{\text{eff}}, H^*)$, $\mathcal{C}(\text{Good}^{\text{eff}}, H^*)$ and $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$ are equivalent. The key idea is the following lemma of Nori.

Lemma 2.2.2. *Let X be an affine k-variety of dimension n and $Z \subseteq X$ is a Zariski closed subset with $\dim(Z) < n$. Then there is a Zariski closed subset Y with $Z \subseteq Y \subseteq X$ and $\dim(Y) < n$ such that (X, Y, n) is a good pair.*

See section 2 of [Nor00] or section 2.5 of [HMS17] for the proof. By using this lemma iteratively, for any affine variety X of dimension n, we can find a filtration

$$\emptyset = F_{-1}X \subset F_0X \subset ... \subset F_{n-1}X \subset F_nX = X$$

such that each $(F_jX, F_{j-1}X, j)$ is very good. Such filtration is called a *very good filtration* of X. The induced chain complex

$$\cdots \to H^i\left(F_iX(\mathbb{C}), F_{i-1}X(\mathbb{C}); \mathbb{Z}\right) \xrightarrow{\delta_i} H^{i+1}\left(F_{i+1}X(\mathbb{C}), F_iX(\mathbb{C}); \mathbb{Z}\right) \to \cdots$$

computes the singular cohomology of X.

Given a very good filtration $F_\bullet X$ on an affine variety X, we denote the chain complex

$$\cdots \to H^i_{\text{Nori}}\left(F_iX, F_{i-1}X\right) \to H^{i+1}_{\text{Nori}}\left(F_{i+1}X, F_iX\right) \to \cdots$$

in $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$, by $\tilde{R}(F_\bullet X)$. Call $\tilde{R}^i(X) \in \mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$ for the i-th cohomology of $\tilde{R}(F_\bullet X)$. Note that $\tilde{R}^i(X)$ is independent of the choice of filtration. Therefore, it defines a functor

$$\tilde{R}^i : \text{Aff} \to \mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$$

such that the singular cohomology realization of $\tilde{R}^i(X)$ is

$$H^*(\tilde{R}^i(X)) = H^i(X(\mathbb{C}), \mathbb{Z}).$$

2.2.2 Equivalence of diagram categories

To show the equivalence of $\mathcal{C}(\text{Pairs}^{\text{eff}}, H^*)$, $\mathcal{C}(\text{Good}^{\text{eff}}, H^*)$ and $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$, we will use corollary 1.2.8, by constructing a representation

$$\text{Pairs}^{\text{eff}} \to \mathcal{C}(\text{VGood}^{\text{eff}}, H^*).$$

For this, we will generalize the idea of $\tilde{R}^i$ using rigidified affine covers.

Definition 2.2.3. Let X be a variety. A *rigidified affine cover* of X is a finite open affine cover $X = \bigcup_{i\in I} U_i$ equipped with a surjective function

$$\begin{aligned} i : X &\twoheadrightarrow I \\ x &\mapsto i_x \end{aligned}$$

such that $x \in U_{i_x}$, for any $x \in X$.

Let $f : X \to Y$ be a morphism of varieties, $X = \bigcup_{i\in I} U_i$ and $Y = \bigcup_{j\in J} V_j$ be rigidified affine covers with rigidifications $i : X \to I$ and $j : Y \to J$. A *morphism* of rigidified covers (over f) is a function $\varphi : I \to J$ such that $f(U_i) \subseteq V_{\varphi(i)}$ and

$$\begin{array}{ccc} X & \xrightarrow{i} & I \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{j} & J \end{array}$$

commutes.

Due to rigidification, if such φ exists, it is unique.

Definition 2.2.4. Let $\mathbb{Z}[\mathrm{Var}]$ be the category with same objects as the category Var and morphisms

$$\mathrm{Mor}_{\mathbb{Z}[\mathrm{Var}]}(X,Y) = \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m}\left\{\sum_k a_k f_k : a_k \in \mathbb{Z}, f_k \in \mathrm{Mor}_{\mathrm{Var}}(X_i, Y_j)\right\}$$

where $X_1, ..., X_n$ and $Y_1, ..., Y_m$ are connected components of X and Y, respectively. It is an additive category with direct sums given by disjoint unions and the zero object is the empty variety $\emptyset$.

We can generalize the definition of rigidification to complexes in $\mathbb{Z}[\mathrm{Var}]$.

Definition 2.2.5. Let $X_\bullet$ be a chain complex in $\mathbb{Z}[\mathrm{Var}]$. An *rigidified affine cover* of $X_\bullet$ is a complex of rigidified affine covers.

Let $X_\bullet \in K^b(\mathbb{Z}[\mathrm{Var}])$. Choose a rigidified affine cover of $X_\bullet$ and a very good filtration on it. This induces a very good filtration $F_\bullet$ on the total complex $T_\bullet$. Let $R(X) := \tilde{R}(F_\bullet T_\bullet)$. This defines a triangulated functor

$$R : K^b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*))$$

such that for every good pair (X, Y, i),

$$H^j(R(\mathrm{Cone}(\mathrm{Y} \to \mathrm{X}))) = \begin{cases} 0, & \text{if } j \neq i \\ H^*(X, Y, i), & \text{if } j = i \end{cases}$$

where

$$H^* : \mathrm{VGood}^{\mathrm{eff}} \to C(\mathrm{VGood}^{\mathrm{eff}}, H^*)).$$

Theorem 2.2.6. *$\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$, $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$ and $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ are equvalent.*

Proof. Let (X, Y, i) be an effective pair. Put

$$H(X, Y, i) := H^i(R(\mathrm{Cone}(\mathrm{Y} \to \mathrm{X}))).$$

H respects edges coming from variety morphisms by construction and it respects edges coming from connecting morphisms since R is triangular. Hence

$$H : \mathrm{Pairs}^{\mathrm{eff}} \to \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$$

is a representation. Clearly,

$$H^* : \mathrm{Pairs}^{\mathrm{eff}} \to \mathbb{Z} - \mathrm{Mod}$$

and the composition

$$\mathrm{Pairs}^{\mathrm{eff}} \xrightarrow{H} \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*) \to \mathbb{Z} - \mathrm{Mod}$$

are isomorphic. The result follows by corollary 1.2.8, □

The discussion above has the following corollary.

Theorem 2.2.7. *There is a natural contravariant triangulated functor*

$$R : K^b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{MM}^{eff}_{Nori})$$

from the bounded homotopy category of $\mathbb{Z}[\mathrm{Var}]$ *to the bounded derived category of* $\mathcal{MM}^{eff}_{Nori}$*, such that for every effective pair* (X, Y, i)*,*

$$H^i(R(\mathrm{Cone}(\mathrm{Y} \to \mathrm{X}))) = H^i_{Nori}(X, Y).$$

2.3 Tannakian Structure

If (X, Y, i) and (X', Y', i') are good pairs, then by Künneth formula,

$$H^n(X \times X', X \times Y' \cup X' \times Y) \simeq \begin{cases} 0 & \text{if } n \neq i + i' \\ H^i(X, Y) \otimes H^{i'}(X', Y') & \text{if } n = i + i'. \end{cases}$$

Motivating from this, we endow a tensor structure to $\mathcal{MM}_{\mathrm{Nori}}$ and this makes it a neutral Tannakian category with the fiber functor H^*. To this end, we define a product structure on $\mathrm{Good}^{\mathrm{eff}}$ and $\mathrm{VGood}^{\mathrm{eff}}$. First note that $\mathrm{Pairs}^{\mathrm{eff}}, \mathrm{Good}^{\mathrm{eff}}$ and

$\mathrm{VGood}^{\mathrm{eff}}$ are graded diagrams with $|(X, Y, i)| = i$. For good (or very good) pairs (X, Y, i) and (X', Y', i'), let

$$(X, Y, i) \times (X', Y', i') = (X \times X', X \times Y' \cup X' \times Y, i + i')$$

with unit $\mathbf{1} = (\operatorname{Spec} k, \emptyset, 0)$. The edges α, β, β' and u are given by natural isomorphism of varieties. By [HMS17, Proposition 9.3.1], this defines a commutative product structure on $\mathrm{Good}^{\mathrm{eff}}$ and $\mathrm{VGood}^{\mathrm{eff}}$, and H^* is a graded multiplicative representation with τ given by Kunneth isomorphism. By proposition 1.3.5, $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$ and $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ are tensor categories and by theorem 2.2.6, so is $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}} = \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$, with tensor functor H^*.

Definition 2.3.1. Let Good (resp. VGood) be the localization of $\mathrm{Good}^{\mathrm{eff}}$ (resp. $\mathrm{VGood}^{\mathrm{eff}}$) with respect to $(\mathbb{G}_m, \{1\}, 1)$.

Good and VGood are also graded diagrams with commutative product structure. Moreover,

$$H^* : \mathrm{Good} \to \mathbb{Z} - \mathrm{Proj}$$

and

$$H^* : \mathrm{VGood} \to \mathbb{Z} - \mathrm{Proj}$$

are graded multiplicative representations. By proposition 1.3.6, $\mathcal{C}(\mathrm{Good}, H^*)$ and $\mathcal{C}(\mathrm{VGood}, H^*)$ are equivalent to the localization of $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}} = \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*) \simeq \mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*) \simeq \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ with respect to $\mathbf{1}(-1)$. Hence $\mathcal{MM}_{\mathrm{Nori}}$, $\mathcal{C}(\mathrm{Good}, H^*)$ and $\mathcal{C}(\mathrm{VGood}, H^*)$ are equivalent. Thus, $\mathcal{MM}_{\mathrm{Nori}}$ is a tensor category with tensor functor H^*.

Definition 2.3.2. Let $n \in \mathbb{Z}$ and $M \in \mathcal{MM}_{\mathrm{Nori}}$.

- $\mathbf{1}(-n) := \mathbf{1}(-1)^{\otimes n}$.
- $M(-n) := M \otimes \mathbf{1}(-n)$.

So far, we have shown that $\mathcal{MM}_{\mathrm{Nori}}$ is an abelian tensor category. The rigidity follows from the following lemma of Nori, Poincare duality for motives. See [HMS17, Lemma 9.3.9] for proof of this.

Lemma 2.3.3. *Let X be a smooth projective of dimension i and $D, E \subseteq X$ divisors such that $D \cup E$ is a normal crossing divisor such that $(X \setminus D, E \setminus (D \cap E), i)$ is a very good pair. Then there is a morphism in $\mathcal{MM}_{Nori}$*

$$q : \mathbf{1} \to H^i_{Nori}(X \setminus D, E \setminus (D \cap E)) \otimes H^i_{Nori}(X \setminus E, D \setminus (D \cap E))(i)$$

such that the dual of $H^*(q)$ realises Poincaré duality.

Combining this lemma with Nori's rigidity criterion ([HMS17, Proposition 8.3.4]), one can conclude that $\mathcal{MM}_{\text{Nori}}$ is rigid. See [HMS17, Theorem 9.3.10] or [Nor00, Theorem 4.4] for detailed proof. Hence we have the following.

Theorem 2.3.4. *$\mathcal{MM}_{Nori}$ is a Tannakian category with the fiber functor H^*. It is equivalent to the category of representations of the faithfully flat affine group scheme over $\mathbb{Z}$*

$$G_{mot}(k, \mathbb{Z}) := \text{Spec}\, A(\text{Good}, H^*).$$

$G_{\text{mot}}(k, \mathbb{Z})$, the Galois group of $\mathcal{MM}_{\text{Nori}}$, is called the *motivic Galois group* in the sense of Nori. Its base change to $\mathbb{Q}$ is denoted by $G_{\text{mot}}(k, \mathbb{Q})$. Fix an algebraic closure $\bar{k} \hookrightarrow \mathbb{C}$. Then there is a short exact sequence

$$1 \to G_{\text{mot}}(\bar{k}, \mathbb{Q}) \to G_{\text{mot}}(k, \mathbb{Q}) \to \text{Gal}(\bar{k}/k) \to 1$$

by [HMS17, Theorem 9.1.16].

2.4 Universality

Finally, we state the universal property of the Nori motives. Expectation from the motives is being the universal Tannakian category for all Weil cohomology theories. As far as is known, the category of Nori motives is universal for all cohomology theories that can be compared with singular cohomology. See [HMS17, Theorem 9.1.10] for its proof.

Theorem 2.4.1. *Let $\mathcal{A}$ be an Abelian category with a faithful exact functor $f : \mathcal{A} \to R-\text{Mod}$ for a Noetherian ring R flat over $\mathbb{Z}$. Let $H'^* : \mathcal{P}(\text{Pairs}^{\text{eff}}) \to \mathcal{A}$ be* *a functor. Let $R \to S$ be a faithfully flat extension. Let $\phi : H_S^* \to (f \circ H'^*)_S$ be an isomorphism of functors, where $-_S$ denotes the composition with $- \otimes S$. Then H'^* extends to $\mathcal{MM}_{Nori}$.*

Example 2.4.2. Take $R = k$, $\mathcal{A} = k - \mathrm{Mod}$, $S = \mathbb{C}$, H'^* algebraic de Rham cohomology and ϕ period isomorphism. Then de Rham cohomology extends to $\mathcal{MM}_{\mathrm{Nori}}$.

Example 2.4.3. Take $R = S = \mathbb{Z}$, $\mathcal{A} = \mathrm{MHS}_{\mathbb{Z}}$, H'^* Hodge realization of a pair and ϕ the map sending a Hodge structure to the underlying $\mathbb{Z}$-module. Then H'^* extends to $\mathcal{MM}_{\mathrm{Nori}}$. This gives Hodge realizations of motives.

Example 2.4.4. Let ℓ be a prime. Take $R = S = \mathbb{Z}_\ell$, $\mathcal{A}$ the category of finitely generated $\mathbb{Z}_\ell$-modules with a continuous operation of $\mathrm{Gal}(\bar{k}/k)$, H'^* ℓ-adic cohomology and ϕ the comparison isomorphism. Then ℓ-adic cohomology extends to $\mathcal{MM}_{\mathrm{Nori}}$.

Let $\sigma, \sigma' : k \to \mathbb{C}$ be two embeddings. Let H^* and H'^* be singular cohomology with respect to σ and σ', respectively. Let $\mathcal{MM}_{\mathrm{Nori}}(\sigma)$ and $\mathcal{MM}_{\mathrm{Nori}}(\sigma')$ be the category of Nori motives constructed using H^* and H'^*, respectively. Using comparison isomorphisms with ℓ-adic cohomology, we can construct a comparison isomorphism between H^* and H'^*. Then by theorem 2.4.1, there is an equivalence of categories $\mathcal{MM}_{\mathrm{Nori}}(\sigma) \to \mathcal{MM}_{\mathrm{Nori}}(\sigma')$. This shows the following.

Corollary 2.4.5. *The category $\mathcal{MM}_{Nori}$ is independent of the choice of embedding $k \to \mathbb{C}$.*

2.5 Computations

The Mayer–Vietoris sequence is motivic. See [Jos16, Corollary 4.5] for its proof.

Theorem 2.5.1. *Let X be a variety and $Y \subseteq X$ be a closed subvariety. Let $\{U, V\}$ be an open cover of X. Then the sequence*

$$\cdots \to H^i_{Nori}(X, Y) \to H^i_{Nori}(U, U \cap Y) \oplus H^i_{Nori}(V, V \cap Y) \\ \to H^i_{Nori}(U \cap V, U \cap V \cap Y) \to H^{i+1}_{Nori}(X, Y) \to \cdots$$

is exact.

Definition 2.5.2. Let $Z \subseteq X$ be a closed immersion with open complement U. We denote

$$H^i_Z(X) := H^i(R(\mathrm{Cone(U \to X)})).$$

By definition, there is a long exact sequence

$$\cdots \to H^i_Z(X) \to H^i_{\mathrm{Nori}}(X) \to H^i_{\mathrm{Nori}}(U) \to H^{i+1}_Z(X) \to \cdots$$

The Gysin sequence is also motivic. See [Jos16, Proposition 5.6] for its proof.

Proposition 2.5.3. *Let X be a smooth, irreducible variety, and let Z be a smooth and irreducible subvariety of codimension c, with open complement U. Then the sequence*

$$\cdots \to H^{i-2c}_{Nori}(Z)(-c) \to H^i_{Nori}(X) \to H^i_{Nori}(U) \to H^{i-2c+1}_{Nori}(Z)(-c) \to \cdots$$

is exact.

The following computations might be useful.

Proposition 2.5.4.

- $H^i_{Nori}(\mathbb{P}^N) = \begin{cases} \mathbf{1}(-n), & \text{if } i = 2n \text{ and } N \geq n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$
- *If Z is a projective variety of dimension n, then $H^{2n}_{Nori}(Z) = \mathbf{1}(-n)$.*
- *If X is a smooth variety and $Z \subset X$ is a smooth, irreducible, closed subvariety of pure codimension n, then $H^{2n}_Z(X) = H^0_{Nori}(Z)(-n) = \mathbf{1}(-n)$.*
- $H^i_{Nori}(\mathbb{A}^N \setminus \{0\}) = \begin{cases} \mathbf{1}(0), & \text{if } i = 0 \\ \mathbf{1}(-N), & \text{if } i = 2N-1 \\ 0, & \text{otherwise.} \end{cases}$
- $H^i_{Nori}(\mathbb{G}^n_m) = \binom{n}{i}\mathbf{1}(-i)$.

See [HMS17, Lemma 9.3.8], for the first three and [Jos16, Proposition 5.4] for the fourth one. The last one follows from the Mayer–Vietoris sequence using open cover $U = \mathbb{G}^{n-1}_m \times (0,1), V = (0,1) \times \mathbb{G}^{n-1}_m$ of $\mathbb{G}^n_m$.

2.6 Weight Filtration

The objects of $\mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$ (the category of mixed Nori motives with rational coefficients) carry a weight filtration compatible with their Hodge realizations.

Definition 2.6.1. A motive $M \in \mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$ is called *pure of weight* $n \in \mathbb{Z}$ if it is a subquotient of $H^{n+2j}_{\mathrm{Nori}}(Y)(j)$ for some Y smooth and projective and $j \in \mathbb{Z}$. A motive is called *pure* if it is a direct sum of pure motives of some weights.

We denote the full subcategory of pure Nori motives by $\mathcal{MM}^{\mathrm{pure}}_{\mathrm{Nori},\mathbb{Q}}$. The following theorem is due to Arapura.

Theorem 2.6.2. *On every motive $M \in \mathcal{MM}_{Nori,\mathbb{Q}}$, there is a unique bounded increasing filtration $(W_nM)_{n\in\mathbb{Z}}$ inducing the weight filtration under the Hodge realization. Moreover, every morphism of Nori motives is strictly compatible with this filtration.*

We call this filtration *weight filtration* and denote $\operatorname{gr}^W_n M := W_nM/W_{n-1}M$.

The original proof is in [Ara13, Theorem 6.3.5]. The idea is to use the spectral sequence of Deligne, giving the weight filtration of the corresponding mixed Hodge structure. See [HMS17, Theorem 10.2.5], for a proof comparing the weight filtration with the geometric motives.

Another proof is given in [Jos16, Theorem 8.2] and the idea is the following. For a pure motive $\widetilde{M}$ of weight n, put

$$0 = W_{n-1}\widetilde{M} \subseteq W_n\widetilde{M} = \widetilde{M}.$$

By [HMS17, Corollary 9.2.23], every motive in $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is a subquotient of a direct sum of motives of the form $H^n_{\mathrm{Nori}}(X, Y)$, where $X = \overline{X} \setminus D$ and $Y = E \setminus (D \cap E)$, with $\overline{X}$ smooth projective and $D, E \subseteq \overline{X}$ closed such that $D \cup E$ is a normal crossing divisor. Call $D = \cup D_i$ and $E = \cup E_i$, with each D_i and E_i smooth. This gives a hypercovering of (X, Y) such that its spectral sequence converges to $H^{p+q}_{\mathrm{Nori}}(X, Y)$. Thus we may assume D and E to be smooth. Finally, consider the exact commutative diagram

$$\begin{array}{ccccc}
H^{n-1}_{\mathrm{Nori}}(E) & \longrightarrow & H^{n}_{\mathrm{Nori}}(\overline{X}, E) & \longrightarrow & H^{n}_{\mathrm{Nori}}(\overline{X}) \\
\downarrow & & \downarrow & & \downarrow \\
H^{n-1}_{\mathrm{Nori}}(Y) & \longrightarrow & H^{n}_{\mathrm{Nori}}(X, Y) & \longrightarrow & H^{n}_{\mathrm{Nori}}(X) \\
\downarrow & & \downarrow & & \downarrow \\
H^{n-2}_{\mathrm{Nori}}(D \cap E)(-1) & \longrightarrow & H^{n-1}_{\mathrm{Nori}}(D, D \cap E)(-1) & \longrightarrow & H^{n-1}_{\mathrm{Nori}}(D)(-1)
\end{array}$$

where rows are from sequences of triples, and columns are from Gysin sequences, both of which are motivic. $H^{n-1}_{\mathrm{Nori}}(E)$ is pure of weight $n-1$, $H^{n}_{\mathrm{Nori}}(\overline{X})$ and $H^{n-2}_{\mathrm{Nori}}(D\cap E)(-1)$ are pure of weight n, and $H^{n-1}_{\mathrm{Nori}}(D)(-1)$ is pure of weight $n+1$. Thus

$$\begin{aligned}
W_{n-2}M &= 0 \\
W_{n-1}M &= \mathrm{Im}(H^{n-1}_{\mathrm{Nori}}(E) \to M) \\
W_{n}M &= \ker(M \to H^{n-1}_{\mathrm{Nori}}(D)(-1)) \\
W_{n+1}M &= M.
\end{aligned}$$

We will give another proof based on the following construction of Deligne described in [Del74], [HMS17, 3.3.4], [GF17, Construction 2.47].

Construction. Let X be a smooth affine variety and $Y \subseteq X$ be a normal crossing divisor, with irreducible components $Y_0, \dots, Y_r$. Let $\dim X = n$. Put

$$\begin{aligned}
D^0 &:= X \\
D^1 &:= Y_0 \sqcup \ldots \sqcup Y_r \\
D^2 &:= (Y_0 \cap Y_1) \sqcup (Y_0 \cap Y_2) \sqcup \ldots \sqcup (Y_{r-1} \cap Y_r) \\
&\vdots \\
D^n &:= (Y_0 \cap \ldots \cap Y_{n-1}) \sqcup \ldots \sqcup (Y_{r-n+1} \cap \ldots \cap Y_r).
\end{aligned}$$

Then the double complex $\Omega^q(D^p)$

$$\begin{array}{ccccc}
\vdots & & & & \\
\uparrow{\scriptstyle d} & & & & \\
\Omega^2(D^0) & \longrightarrow & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle -d} & & \\
\Omega^1(D^0) & \longrightarrow & \Omega^1(D^1) & \longrightarrow & \vdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle -d} & & \uparrow{\scriptstyle d} \\
\Omega^0(D^0) & \longrightarrow & \Omega^0(D^1) & \longrightarrow & \Omega^0(D^2)
\end{array}$$

computes the relative De Rham cohomology $H^\bullet_{\mathrm{dR}}(X, Y)$. Here the horizontal maps are given by $\bigoplus \epsilon(I, J)\varphi_{IJ}$, where $\varphi_{IJ} : \Omega^q(\bigcap_{i\in I} Y_i) \to \Omega^q(\bigcap_{j\in J} Y_j)$ are the restriction maps and ϵ is defined as follows. Put $J = \{j_1, \dots, j_p\}$ with $j_1 < \dots < j_p$. If $I = J \setminus \{j_s\}$, then $\epsilon(I, J) = (-1)^s$, otherwise $\epsilon(I, J) = 0$.

Example 2.6.3. Let $a, b \notin \{0, 1\}$. Let $B = M_0 \cup M_1 \cup M_2 \subseteq \mathbb{P}^2_{\mathbb{C}}$, where

$$M_0 : bz_0 = z_1$$
$$M_1 : z_0 = z_1 + z_2$$
$$M_2 : az_0 = z_2.$$

We will calculate $H^2_{\mathrm{dR}}(\mathbb{G}^2_m, B \cap \mathbb{G}^2_m)$. Call $X = \mathbb{G}^2_m$ and $Y = B \cap \mathbb{G}^2_m$. We may write

$$X = \operatorname{Spec} \mathbb{C}[x, y, \frac{1}{x}, \frac{1}{y}]$$

and $Y = Y_0 \cup Y_1 \cup Y_2$, where $Y_i = M_i \cap X$, so

$$Y_0 = \operatorname{Spec} \mathbb{C}[y, \frac{1}{y}], \quad Y_1 = \operatorname{Spec} \mathbb{C}[z, \frac{1}{z}, \frac{1}{1-z}], \quad Y_2 = \operatorname{Spec} \mathbb{C}[x, \frac{1}{x}]$$

and

$$Y_0 \cap Y_1 = \{(b, 1-b)\}, \quad Y_0 \cap Y_2 = \{(b, a)\}, \quad Y_1 \cap Y_2 = \{(1-a, a)\}.$$

We will consider the following double complex.

$$
\begin{array}{ccccc}
\mathbb{C}[x,y,\frac{1}{x},\frac{1}{y}]dx\wedge dy & & & & \\
\uparrow d & & & & \\
\mathbb{C}[x,y,\frac{1}{x},\frac{1}{y}]dx\oplus\mathbb{C}[x,y,\frac{1}{x},\frac{1}{y}]dy & \longrightarrow & \mathbb{C}[x,\frac{1}{x}]dx\oplus\mathbb{C}[y,\frac{1}{y}]dy\oplus\mathbb{C}[z,\frac{1}{z},\frac{1}{1-z}]dz & & \\
\uparrow d & & \uparrow -d & & \\
\mathbb{C}[x,y,\frac{1}{x},\frac{1}{y}] & \xrightarrow{\varphi} & \mathbb{C}[x,\frac{1}{x}]\oplus\mathbb{C}[y,\frac{1}{y}]\oplus\mathbb{C}[z,\frac{1}{z},\frac{1}{1-z}] & \xrightarrow{\psi} & \mathbb{C}^{\oplus 3}
\end{array}
$$

The first page of the spectral sequence reads

$$
\begin{array}{ccccc}
\langle\frac{dx}{x}\wedge\frac{dy}{y}\rangle & & & & \\
\langle\frac{dx}{x},\frac{dy}{y}\rangle & \xrightarrow{\xi_1} & \langle\frac{dx}{x},\frac{dy}{y},\frac{dz}{z},\frac{dz}{1-z}\rangle & & \\
\mathbb{C} & \xrightarrow{\varphi_1} & \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C} & \xrightarrow{\psi_1} & \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}
\end{array}
$$

where $\varphi_1:\alpha\mapsto(\alpha,\alpha,\alpha)$, $\psi_1:(\alpha,\beta,\theta)\mapsto(\beta-\alpha,\theta-\alpha,\theta-\beta)$ and

$$
\begin{aligned}
\xi_1:\frac{dx}{x}&\mapsto\frac{dx}{x}+\frac{dz}{z}\\
\frac{dy}{y}&\mapsto\frac{dy}{y}-\frac{dz}{1-z}.
\end{aligned}
$$

Finally, the second page reads

$$
\begin{array}{ccc}
\mathbb{C} & & \\
0 & \mathbb{C}\oplus\mathbb{C} & \\
0 & 0 & \mathbb{C}
\end{array}
$$

and therefore $H^2_{\mathrm{dR}}(X,Y)=\mathbb{C}^4=\langle\frac{dx}{x}\wedge\frac{dy}{y},\frac{dx}{x},\frac{dy}{y},(1,0,0)\rangle$.

Proof of theorem 2.6.2. It suffices to define the weight filtration for effective motives, then the general case will follow by localization. By [HMS17, Corollary 9.2.25], every motive in $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is a subquotient of a direct sum of motives of the form $H^n_{\mathrm{Nori}}(X,Y)$, where X is smooth affine and Y is a normal crossing divisor. Hence it suffices to define the weight filtration for motives $M=H^n_{\mathrm{Nori}}(X,Y)$ of this form.

Let $F_\bullet X$ be a very good filtration of X. It also induces a very good filtration $F_\bullet D$ on any subvariety $D\subseteq X$ by $F_qD=X_q\cap D$. Let D^p be as in the construction above. Define the weight filtration of M as the filtration coming from the spectral sequence of double complex $F_q(D^p)$. □

2.6.1 Mixed Tate motives

A mixed Tate object of $\mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$ is called a *mixed Tate Nori motive* and we denote the full subcategory of these objects by $\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}$. Then $(\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}, \mathbf{1}(1))$ is a mixed Tate category, in the sense of Section 1.1. The subcategory $\mathcal{TM}_{\mathrm{Nori},\mathbb{Q}}$ of pure Tate Nori motives is equivalent to the category of graded $\mathbb{Q}$-vector spaces.

Example 2.6.4. Let $B \subseteq \mathbb{G}_m$ be a divisor, with $|B| = r$. (In particular if $r = 2$, then $(\mathbb{G}_m; B) \in A_1$.) We will find the weight structure of $H^1_{\mathrm{Nori}}(\mathbb{G}_m, B)$. We have the following exact sequence

$$0 \to \mathbf{1}(0) \to r \cdot \mathbf{1}(0) \to H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) \to \mathbf{1}(-1) \to 0$$

since $H^0_{\mathrm{Nori}}(B) = r \cdot \mathbf{1}(0)$. Then,

$$\begin{aligned} W_{-1}H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) &= 0 \\ W_0 H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) &= W_1 H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) = (r-1) \cdot \mathbf{1}(0) \\ W_2 H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) &= H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) \end{aligned}$$

and

$$\begin{aligned} \mathrm{gr}^W_0 H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) &= (r-1) \cdot \mathbf{1}(0) \\ \mathrm{gr}^W_2 H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) &= \mathbf{1}(-1) \end{aligned}$$

Therefore $H^1_{\mathrm{Nori}}(\mathbb{G}_m, B)$ is mixed Tate.

Example 2.6.5. Let $a \neq 0, 1$. Let $B = M_0 \cup M_1 \cup M_2 \subseteq \mathbb{P}^2_{\mathbb{C}}$, where

$$\begin{aligned} M_0 &: z_0 = z_1 \\ M_1 &: z_0 = z_1 + z_2 \\ M_2 &: az_0 = z_2. \end{aligned}$$

We will find the weight structure of $H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right)$. Consider the exact sequence

$$0 \to 2\cdot\mathbf{1}(-1) \to H^1_{\text{Nori}}\left(B\cap\mathbb{G}_m^2\right) \to H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) \to \mathbf{1}(-2) \to 0.$$

Since

$$\begin{aligned}
\operatorname{gr}_0^W H^1_{\text{Nori}}\left(B\cap\mathbb{G}_m^2\right) &= \mathbf{1}(0)\\
\operatorname{gr}_2^W H^1_{\text{Nori}}\left(B\cap\mathbb{G}_m^2\right) &= 3\cdot\mathbf{1}(-1)\\
\operatorname{gr}_4^W H^1_{\text{Nori}}\left(B\cap\mathbb{G}_m^2\right) &= 0,
\end{aligned}$$

we have

$$\begin{aligned}
\operatorname{gr}_0^W H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) &= \mathbf{1}(0)\\
\operatorname{gr}_2^W H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) &= \mathbf{1}(-1)\\
\operatorname{gr}_4^W H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) &= \mathbf{1}(-2).
\end{aligned}$$

So, $H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right)$ is also mixed Tate.

Example 2.6.6. In general, let $B = M_0\cup M_1\cup M_2 \subseteq \mathbb{P}^2_{\mathbb{C}}$, where M_i are lines in $\mathbb{P}^2_{\mathbb{C}}$ not in general position and they are not axis lines $z_i = 0$. Then

$$\begin{aligned}
\operatorname{gr}_0^W H^1_{\text{Nori}}\left(B\cap\mathbb{G}_m^2\right) &= \mathbf{1}(0)\\
\operatorname{gr}_2^W H^1_{\text{Nori}}\left(B\cap\mathbb{G}_m^2\right) &= s\cdot\mathbf{1}(-1),
\end{aligned}$$

where s is the number of intersection of B with $z_1 = 0$ and $z_2 = 0$. Using the exact sequence

$$0 \to 2\cdot\mathbf{1}(-1) \to H^1_{\text{Nori}}\left(B\cap\mathbb{G}_m^2\right) \to H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) \to \mathbf{1}(-2) \to 0.$$

we have

$$\begin{aligned}
\operatorname{gr}_0^W H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) &= \mathbf{1}(0)\\
\operatorname{gr}_2^W H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) &= (s-2)\cdot\mathbf{1}(-1)\\
\operatorname{gr}_4^W H^2_{\text{Nori}}\left(\mathbb{G}_m^2, B\cap\mathbb{G}_m^2\right) &= \mathbf{1}(-2)
\end{aligned}$$

and $s-2 \in \{0,1,2,3,4\}$. Here $\mathbf{1}(-2)$ is coming from the torus $\mathbb{G}_m^2$ and $\mathbf{1}(0)$ is coming from the triangle defined by B.

Chapter 3

AOMOTO POLYLOGARITHMS

3.1 Classical Polylogarithms

There is a generalization of logarithms called *polylogarithms* which is defined inductively by $\ell i_1(z) = -\log(1-z)$ and

$$d\ell i_n(z) = \ell i_{n-1}(z)\frac{dz}{z},$$

with $\ell i_n(0) = 0$. They have the power series expansion

$$\ell i_n(z) = \sum_{1 \leq m} \frac{z^m}{m^n},$$

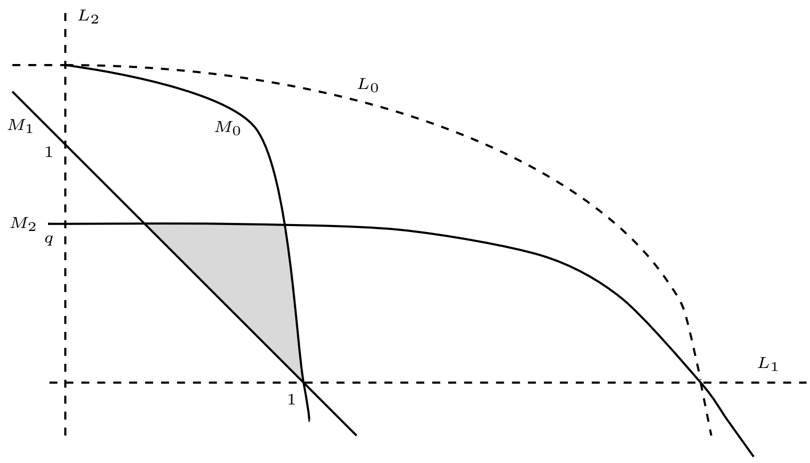
for $|z| < 1$. The polylogarithms of rational numbers are also periods. They come from the following pairs. Fix some $q \in \mathbb{Q} \setminus \{0,1\}$. Let z_i, $i = 0, 1, ..., n$, be the homogeneous coordinates on $\mathbb{P}^n_{\mathbb{Q}}$. Let L_i be the hyperplanes defined by $z_i = 0$ and M_i be the hyperlanes defined as $M_0 : z_0 = z_1; M_1 : z_0 = z_1 + z_2; M_i : z_i = z_{i+1}$ for $2 \leq i < n$; and $M_n : qz_0 = z_n$. Let $M_q = \bigcup M_i$ and $L = \bigcup L_i$. Then $\ell i_n(q)$ appears in the period matrix of $(\mathbb{P}^n_{\mathbb{Q}} \setminus L, M_q \setminus (L \cap M_q))$ on the top cohomology. We will call the configuration (L, M_q) as the *polylogarithmic configuration* of q.

ℓi_2 is called *dilogarithm*. Its configuration is given by

$$M_0 : z_0 = z_1$$

$$M_1 : z_0 = z_1 + z_2$$

$$M_2 : qz_0 = z_2.$$



Call $\mathcal{D}(q)$ for the triangle given by M_i. Then

$$\ell i_2(q) = \int_{\mathcal{D}(q)} \frac{dx}{x} \wedge \frac{dy}{y}.$$

The problem with

$$\ell i_2(z) = -\int_0^z \log(1-t)\frac{dt}{t}$$

is that its analytic continuation to $\mathbb{C} \setminus [1, \infty)$ is multivalued since it jumps by $2\pi \mathrm{i} \log|z|$ as z crosses the cut. Defining the Bloch-Wigner dilogarithm

$$D(z) = \mathrm{Im}(\ell i_2(z)) + \arg(1-z)\log|z|$$

solves the problem. This is the 2-dimensional analogue of $\log(z)$ and $\log|z|$.

There is a 5-term relation of dilagoraithm

$$\begin{aligned}&\ell i_2(x) - \ell i_2(y) + \ell i_2(y/x) - \ell i_2((1-x^{-1})/(1-y^{-1})) + \ell i_2((1-x)/(1-y))\\ =&\pi^2/6 - \log(x)\log((1-x)/(1-y))\end{aligned}$$

which is zero modulo elementary functions. Passing to the Bloch-Wigner dilogarithm, this becomes

$$D(x) - D(y) + D(y/x) - D((1-x^{-1})/(1-y^{-1})) + D((1-x)/(1-y)) = 0.$$

This implies that $D(x^{-1}) = -D(x)$ and $D(1-x) = -D(x)$, therefore the 6-fold symmetry

$$D(x) = -D(1/x) = D(1-1/x) = -D(x/(x-1)) = D(1/(1-x)) = -D(1-x).$$

The 5-term relation can be even better using cross-ratio. If we define

$$\begin{aligned}\widetilde{D}\left(z_0, z_1, z_2, z_3\right) &= D\left(r\left(z_0, z_1, z_2, z_3, z_4\right)\right) \\ &= D\left(\frac{z_0 - z_3}{z_0 - z_2}\frac{z_1 - z_2}{z_1 - z_3}\right)\end{aligned}$$

then the 5-term relation of the Bloch-Wigner dilogarithm is equivalent to

$$\sum_{i=0}^{4}(-1)^i \widetilde{D}\left(z_0, \ldots, \widehat{z_i}, \ldots, z_4\right) = 0$$

since

$$\begin{aligned}\widetilde{D}(0, \infty, 1, x) &= D(x) \\ \widetilde{D}(0, \infty, 1, y) &= D(y) \\ \widetilde{D}(0, \infty, x, y) &= D(y/x) \\ \widetilde{D}(0, 1, x, y) &= D((1 - x^{-1})/(1 - y^{-1})) \\ \widetilde{D}(\infty, 1, x, y) &= D((1 - x)/(1 - y)).\end{aligned}$$

It turns out that the 5-term relation of the Bloch-Wigner dilogarithm suffices to give all relations of the finite sum form $\sum D(x_i(t)) = C$, where x_i are rational functions over $\mathbb{C}$ and $C \in \mathbb{C}$ is a constant. This motivates the following definition.

Definition 3.1.1. Let A be an Artinian local ring with infinite residue field. The *Bloch group* $B_2(A)$ is defined as quotient of the free abelian group generated by $[x]$, for $x, (1 - x) \in A^\times$, by the subgroup generated by elements of the form

$$[x] - [y] + [y/x] - [(1 - x^{-1})/(1 - y^{-1})] + [(1 - x)/(1 - y)]$$

for all $x, y \in A^\times$ with $(1 - x), (1 - y), (1 - x/y) \in A^\times$.

The map

$$\begin{aligned}\delta : B_2(A) &\to \Lambda^2_{\mathbb{Z}} A^\times \\ [x] &\mapsto (1 - x) \wedge x\end{aligned}$$

plays a crucial role in motivic cohomology. We prove the 5-term relation following [Gon95].

Proposition 3.1.2. *We have*

$$\sum_{i=0}^{4}(-1)^i \widetilde{D}\left(z_0, \ldots, \widehat{z_i}, \ldots, z_4\right) = 0$$

for distinct $z_i \in \mathbb{P}^1_{\mathbb{C}}$.

Proof. Let k be a field. Let $C_m(n)$ (respectively $C_m(\mathbb{P}^n_k)$) be the free abelian group generated by configurations $(z_0, ..., z_{m-1})$ of m vectors in k^n (respectively m points in $\mathbb{P}^n_k$) in generic position (i.e., every $m_0 \leq m$ of them generate a m_0-dimensional subspace). Let

$$\begin{aligned} \delta : C_5(\mathbb{P}^1_k) &\to C_4(\mathbb{P}^1_k) \\ (z_0, ..., z_4) &\mapsto \sum_{i=0}^{4}(-1)^i \left(z_0, \ldots, \widehat{z_i}, \ldots, z_4\right) \end{aligned}$$

and

$$\begin{aligned} \delta : C_4(\mathbb{P}^1_k) \xrightarrow{r} k^\times &\xrightarrow{\delta} \Lambda^2 k^\times \\ z &\mapsto (1-z) \wedge z. \end{aligned}$$

We claim that the composition $C_5(\mathbb{P}^1_k) \xrightarrow{\delta} C_4(\mathbb{P}^1_k) \xrightarrow{\delta} \Lambda^2 k^\times$ is zero. First, we will assume this and prove the result.

We want to show that, for $k = \mathbb{C}$, we have $\widetilde{D}(\delta(z_0, ..., z_4)) = 0$. Consider $\widetilde{D} \circ \delta = D \circ r \circ \delta$ as a function on the manifold of configurations of 5 points in $\mathbb{P}^1_{\mathbb{C}}$ (this is a submanifold of $(\mathbb{P}^1_{\mathbb{C}})^5$). We will show that $d(D(r(\delta(z_0, ..., z_4)))) = 0$. This will imply that $D \circ r \circ \delta$ is constant. Since

$$(D \circ r \circ \delta)(x, x, y, y, z) = (D \circ r)(x, x, y, y) = D(1) = 0,$$

we have $D \circ r \circ \delta = 0$.

Since $\log(z) = \log|z| + \mathrm{i}\arg(z)$, we have $D(z) = \mathrm{Im}(\ell i_2(z) + \log(1-z)\log|z|)$. Then,

$$\begin{aligned} dD(z) &= \mathrm{Im}(-\log(1-z)d\log z + \log(1-z)d\log|z| + \log|z|d\log(1-z)) \\ &= -\log|1-z|d\arg z + \log|z|d\arg(1-z). \end{aligned}$$

Thus $dD : \mathbb{C}^\times \to \mathbb{C}$ factors as

$$\begin{aligned} \mathbb{C}^\times \xrightarrow{\delta} \Lambda^2\mathbb{C}^\times &\to \mathbb{C} \\ x \wedge y &\mapsto -\log|x| d\arg y + \log|y| d\arg(x). \end{aligned}$$

Therefore,

$$\begin{aligned} dD \circ r \circ \delta : & C_5(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\delta} C_4(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{r} \mathbb{C}^\times \xrightarrow{dD} \mathbb{C} \\ = & C_5(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\delta} C_4(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{r} \mathbb{C}^\times \xrightarrow{\delta} \Lambda^2\mathbb{C}^\times \to \mathbb{C} \\ = & C_5(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\delta} C_4(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\delta} \Lambda^2\mathbb{C}^\times \to \mathbb{C} \end{aligned}$$

is zero.

To show that $C_5(\mathbb{P}^1_k) \xrightarrow{\delta} C_4(\mathbb{P}^1_k) \xrightarrow{\delta} \Lambda^2 k^\times$ is zero, we will construct the following commutative diagram.

$$\begin{array}{ccccc} C_5(2) & \xrightarrow{d} & C_4(2) & \xrightarrow{d} & C_3(2) \\ \downarrow f_2^{(2)} & & \downarrow f_1^{(2)} & & \downarrow f_0^{(2)} \\ C_5(\mathbb{P}^1_k) & \xrightarrow{\delta} & C_4(\mathbb{P}^1_k) & \xrightarrow{\delta} & \Lambda^2 k^\times \\ \downarrow & & \downarrow r & & \downarrow \mathrm{id} \\ 0 & \longrightarrow & B_2(k) & \xrightarrow{\delta} & \Lambda^2 k^\times \end{array}$$

such that $d \circ d = 0$, where $f_2^{(2)}$ and $f_1^{(2)}$ are projectivization maps. Commutativity of the first two rows and surjectivity of $f_2^{(2)}$ imply that $\delta \circ \delta = 0$.

We define the Grassmannian complex as

$$C_5(2) \xrightarrow{d} C_4(2) \xrightarrow{d} C_3(2)$$

where

$$d : (z_0, ..., z_m) \mapsto \sum_{i=0}^{m} (-1)^i \left(z_0, \ldots, \widehat{z_i}, \ldots, z_m\right).$$

Then $d \circ d = 0$.

Let ω be a volume form in k^2. Set $\Delta(z_1, z_2) := \langle \omega, z_1 \wedge z_2 \rangle$, for $z_i \in k^2$. Define

$$\begin{aligned} f_0^{(2)} : C_3(2) \to & \Lambda^2 k^\times \\ (z_0, z_1, z_2) \mapsto & \Delta(z_0, z_1) \wedge \Delta(z_0, z_2) \\ & -\Delta(z_0, z_1) \wedge \Delta(z_1, z_2) \\ & +\Delta(z_0, z_2) \wedge \Delta(z_1, z_2). \end{aligned}$$

The RHS is $\frac{\Delta(z_0,z_1)}{\Delta(z_0,z_2)} \wedge \frac{\Delta(z_0,z_2)}{\Delta(z_1,z_2)}$ modulo 2-torsion by adding $-\Delta(z_0, z_2) \wedge \Delta(z_0, z_2)$. Let $\{e_1, e_2\}$ be a basis for k^2. Then $\omega = a \cdot e_1 \wedge e_2$ for some $a \in k^\times$. If

$$z_1 = c_{11}e_1 + c_{21}e_2$$
$$z_2 = c_{21}e_1 + c_{22}e_2,$$

then

$$\Delta(z_1, z_2) = \Delta(c_{11}e_1 + c_{21}e_2, c_{21}e_1 + c_{22}e_2) = a \cdot \det \begin{pmatrix} \langle e_1, z_1\rangle & \langle e_1, z_2\rangle \\ \langle e_2, z_1\rangle & \langle e_2, z_2\rangle \end{pmatrix}$$
$$= a \cdot \det \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = a \cdot \det \begin{pmatrix} z_1 & z_2 \end{pmatrix}.$$

Let us assume $\operatorname{char} k \neq 2$. Then $f_0^{(2)}$ is independent of the choice of e_1, e_2, a and therefore of ω. Also, by direct computation,

$$f_0^{(2)} \circ d : (z_0, z_1, z_2, z_3) \mapsto \frac{\Delta(z_0, z_1)\Delta(z_2, z_3)}{\Delta(z_0, z_2)\Delta(z_1, z_3)} \wedge \frac{\Delta(z_0, z_3)\Delta(z_1, z_2)}{\Delta(z_0, z_2)\Delta(z_1, z_3)}$$

modulo 2-torsion, so the map

$$C_4(2) \xrightarrow{d} C_3(2) \xrightarrow{f_0^{(2)}} \Lambda^2 k^\times$$

does not depend on the lengths of the vectors z_i. Therefore it factors through the projectivization $f_1^{(2)} : C_4(2) \to C_4(\mathbb{P}^1_k)$, i.e.

$$\begin{array}{ccc} C_4(2) & \xrightarrow{d} & C_3(2) \\ \big\downarrow f_1^{(2)} & & \big\downarrow f_0^{(2)} \\ C_4(\mathbb{P}^1_k) & \overset{\delta'}{\dashrightarrow} & \Lambda^2 k^\times \end{array}$$

commutes for some δ'.

Finally, by direct computation, we will show that $\delta' = \delta$. Since $f_0^{(2)}$ is independent of the choice of ω, fix $\omega = (1, 0) \wedge (0, 1)$. Let $\bar{a} = (a : 1), \bar{b} = (b : 1) \in \mathbb{P}^1_k$ be arbitrary. Also denote $\infty = (1 : 0) \in \mathbb{P}^1_k$. Then

$$\Delta((a, 1), (b, 1)) = a - b$$
$$\Delta((a, 1), (1, 0)) = -1$$
$$\Delta((1, 0), (a, 1)) = 1.$$

So, denoting $\bar{z}_i$ for the projectivization of $z_i \in k^2$,

$$\delta' : (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3) \mapsto \frac{\Delta(z_0, z_1)\Delta(z_2, z_3)}{\Delta(z_0, z_2)\Delta(z_1, z_3)} \wedge \frac{\Delta(z_0, z_3)\Delta(z_1, z_2)}{\Delta(z_0, z_2)\Delta(z_1, z_3)}$$
$$= (1 - z) \wedge z$$

where $z = r(\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)$. □

See [Zag07] for more information about dilogarithms, and see [Ünv21] for the infinitesimal version of this theory.

3.2 Aomoto Polylogarithms

Let k be a field. We call an n-*simplex* a family of $n + 1$ hyperplanes $(L_0, ..., L_n)$ of $\mathbb{P}^n = \mathbb{P}^n_k$. A pair of simplices (L, M) is said to be *admissible* if they do not have a common face. Define $A_n(k)$ (or just A_n) as the abelian group generated by $(L; M)$ where (L, M) is an admissible pair of simplices in $\mathbb{P}^n$ subject to the following relations:

1. If the hyperplanes of one of L or M are not in general position (i.e. degenerate), then $(L; M) = 0$.

2. For every $\sigma \in S_n$,

$$(\sigma L; M) = (L; \sigma M) = (-1)^{|\sigma|}(L; M)$$

   where σL and σM, are defined by the natural action of S_n on a set indexed by $1, ..., n$.

3. For every family of hyperplanes $L_0, ..., L_{n+1}$ and an n-simplex M,

$$\sum (-1)^j (\hat{L}^j; M) = 0,$$

   where $\hat{L}^j = (L_0, ..., \hat{L}_j, ..., L_{n+1})$. Similarly, for every family of hyperplanes $M_0, ..., M_{n+1}$ and an n-simplex L,

$$\sum (-1)^j (L; \hat{M}^j) = 0,$$

   where $\hat{M}^j = (M_0, ..., \hat{M}_j, ..., M_{n+1})$.

4. For every $g \in \mathrm{PGL}_{n+1}(k)$,

$$(gL; gM) = (L; M).$$

Put $A_0 = \mathbb{Z}$. The cross-ratio of 4 different points on $\mathbb{P}^1$ defines an isomorpism $A_1 \xrightarrow{\sim} k^\times$.

In the case of $k = \mathbb{C}$, we may attach a period $a(X)$ to each element $X \in A_n(\mathbb{C})$ in the following way. Let $(L; M)$ be a non-zero generator of $A_n(\mathbb{C})$. Define

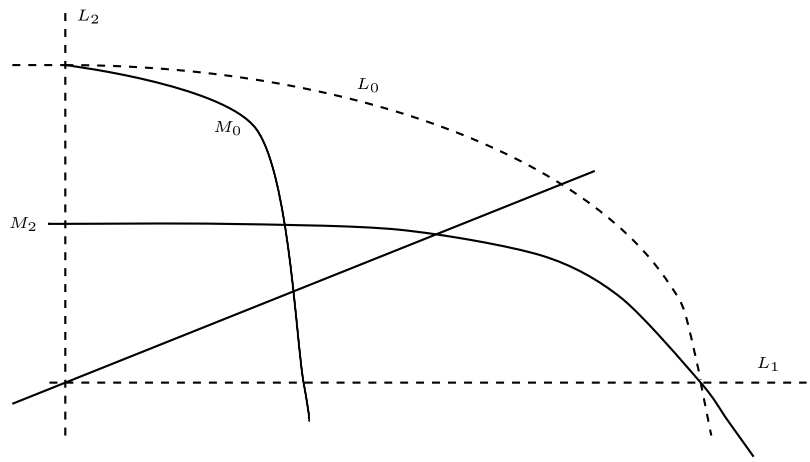
$$a(L, M) = \int_{\Delta_M} \omega_L$$

where Δ_M is the n-simplex given by M and $\omega_L = d\log(z_1/z_0) \wedge ... \wedge d\log(z_n/z_0)$ given that $z_i = 0$ is a homogeneous equation of L_i.

The multiplication map $\mu : A_{n'} \times A_{n''} \to A_n$, for $n' + n'' = n$, is defined on the generators in the following way. Let (L', M') and (L'', M'') be two admissible pairs of non-degenerate simplices from $\mathbb{P}^{n'}$ and $\mathbb{P}^{n''}$, respectively. Also, let L be a non-degenerate simplex from $\mathbb{P}^n$. Identify the affine spaces $\mathbb{P}^n \setminus L_0$ and $(\mathbb{P}^{n'} \setminus L'_0) \times (\mathbb{P}^{n''} \setminus L''_0)$. Then $M' \times M''$ can be seen in $\mathbb{P}^n \setminus L_0$ and hence in $\mathbb{P}^n$. Cutting this product into simplices in $\mathbb{P}^n$ defines an element in A_n which is defined as the product of $(L'; M')$ and $(L''; M'')$. We call an element of A_n a *prism* if it is in the image of any multiplication map μ from the lower degrees. The subgroup generated by all prisms in A_n is denoted by P_n. [BGSV07] also defines a comultiplication $A_n \to A_{n'} \otimes A_{n''}$ compatible with the multiplication. As a result, $A = \bigoplus A_n$ is a commutative Hopf algebra.

3.2.1 Aomoto dilogarithms

For $\alpha \in k^\times$, denote the polylogarithmic configuration (L, M_α) (as defined in introduction) of α in $\mathbb{P}^2_k$ by $l_2(\alpha) := (L; M_\alpha)$. Call a *half-square* an admissible pair of triangles as below. Let Σ be the subgroup of A_2 generated by the half-squares and $l_2(1)$.



Recall that $B_2(k)$ is the quotient of the abelian group generated by $[x]$, for $x \in k \setminus \{0, 1\}$, by the subgroup generated by elements of the form

$$[x] - [y] + [y/x] - [(1 - x^{-1})/(1 - y^{-1})] + [(1 - x)/(1 - y)]$$

for all $x, y \in k \setminus \{0, 1\}$ with $x \neq y$. The second main theorem of [BGSV07] relates this group with A_2, as expected. We state and prove this theorem following that article.

Theorem 3.2.1. *The map*

$$l_2 : B_2(k) \to A_2(k)/\Sigma$$
$$[\alpha] \mapsto l_2(\alpha)$$

is an isomorphism.

Proof. We define $B_2'(k)$, also known as the Bloch group, motivated by the 5-term relation of $\widetilde{D} = D \circ r$. Let $B_2'(k)$ be the abelian group generated by

$$\{(x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{P}^1_k\}$$

subject to the following relations.

1. $(x_1, x_2, x_3, x_4) = 0$ if $x_i = x_j$ for some $i \neq j$.

2. For every $\sigma \in S_4$,

$$(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) = (-1)^{|\sigma|}(x_1, x_2, x_3, x_4).$$

3. For every $x_1, x_2, x_3, x_4, x_5 \in \mathbb{P}^1_k$,

$$\sum(-1)^j(x_1, ..., \hat{x}_j, ..., x_5) = 0.$$

4. For every $g \in \mathrm{PGL}_2(k)$,

$$(gx_1, gx_2, gx_3, gx_4) = (x_1, x_2, x_3, x_4).$$

Define $r : B_2' \to B_2$ by the cross-ratio. This gives an isomorphism with the inverse map

$$\begin{aligned} B_2 &\to B_2' \\ [\alpha] &\mapsto (0, \infty, 1, \alpha). \end{aligned}$$

We will construct a map $\eta : A_2/\Sigma \to B_2'$ such that the compositions

$$B_2 \xrightarrow{l_2} A_2/\Sigma \xrightarrow{\eta} B_2' \xrightarrow{r} B_2$$

$$A_2/\Sigma \xrightarrow{\eta} B_2' \xrightarrow{r} B_2 \xrightarrow{l_2} A_2/\Sigma$$

are identity. Let F be the free abelian group generated by

$$\{(m, M) \mid M \subset \mathbb{P}^2 \text{ is a straight line, } m \in M \text{ is a point}\}.$$

For a triangle $M = (M_0, M_1, M_2)$ define

$$\begin{aligned} \psi(M) :=&(M_0 \cap M_1, M_1) - (M_0 \cap M_1, M_0) \\ &+(M_1 \cap M_2, M_2) - (M_1 \cap M_2, M_1) \\ &+(M_0 \cap M_2, M_0) - (M_0 \cap M_2, M_2). \end{aligned}$$

Also for a triangle $L = (L_0, L_1, L_2)$ define a homomorphism $\rho_L : F \to B_2'$ by

$$\rho_L(m, M) = (M \cap L_0, M \cap L_1, M \cap L_2, m)$$

if M does not coincide with a side of L and $\rho_L(m, M) = 0$ otherwise. Finally, define

$$\begin{aligned} \eta : A_2/\Sigma &\to B_2' \\ (L; M) &\mapsto \rho_L(\psi(M)). \end{aligned}$$

By direct calculation, $l_2(1)$ and half-squares vanish under η.

η maps the dilogarithmic configuration

$$\begin{aligned} &M_0 : z_0 = z_1 \\ &M_1 : z_0 = z_1 + z_2 \\ &M_2 : \alpha z_0 = z_2. \end{aligned}$$

of α to

$$\rho_L((M_{12}, M_1)) = ((0:1:-1),(1:0:1),(1:1:0),(1:1-\alpha:\alpha))$$

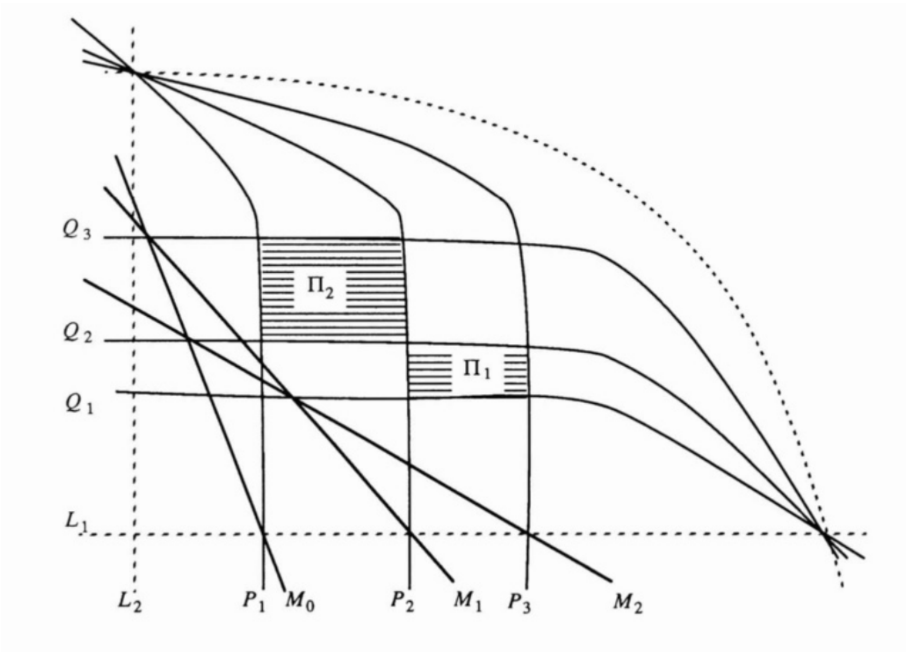
as the other terms vanish and therefore

$$\begin{aligned} r\eta l_2([\alpha]) &= r((0:1),(1:0),(1:1),(1:1-\alpha)) \\ &= [1/(1-\alpha)] \\ &= [\alpha]. \end{aligned}$$

On the other hand

$$\begin{aligned} \eta((L;M)) = &(M_1 \cap L_0, M_1 \cap L_1, M_1 \cap L_2, M_0 \cap M_1) \\ &- (M_0 \cap L_0, M_0 \cap L_1, M_0 \cap L_2, M_0 \cap M_1) \\ &+ (M_2 \cap L_0, M_2 \cap L_1, M_2 \cap L_2, M_1 \cap M_2) \\ &- (M_1 \cap L_0, M_1 \cap L_1, M_1 \cap L_2, M_1 \cap M_2) \\ &+ (M_0 \cap L_0, M_0 \cap L_1, M_0 \cap L_2, M_0 \cap M_2) \\ &- (M_2 \cap L_0, M_2 \cap L_1, M_2 \cap L_2, M_0 \cap M_2). \end{aligned}$$

Assuming these triangles are in general position, the picture



(taken from [BGSV07, Figure 3.2]) shows that

$$\begin{aligned}(M_0, M_1, M_2) = &- (M_1, Q_3, P_2) + (M_0, Q_3, P_1)\\ &- (M_2, Q_1, P_3) + (M_1, Q_1, P_2)\\ &- (M_0, Q_2, P_1) + (M_2, Q_2, P_3)\\ &- \Pi_1 + \Pi_2.\end{aligned}$$

Thus

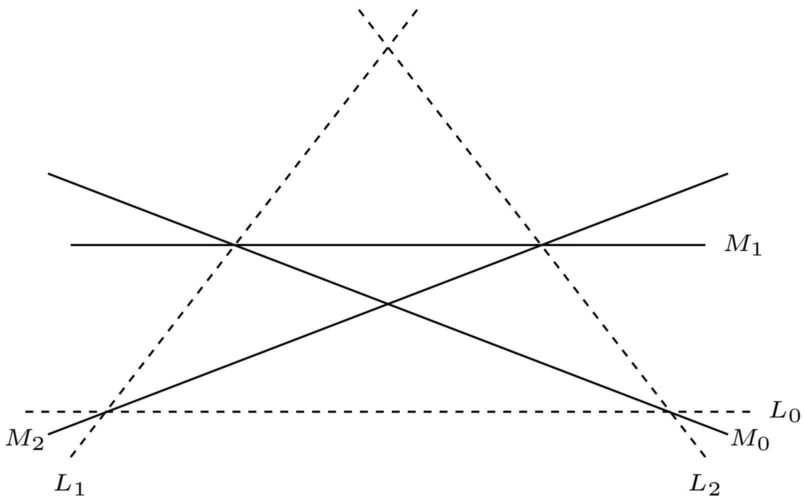
$$(L; M) - l_2 r\eta((L; M)) = -(L; \Pi_1) + (L; \Pi_2) \in P_2 \leqslant \Sigma.$$

For the well-definedness of l_2, see Proposition 3.11 of [Zha01]. □

Corollary 3.2.2. *An Aomoto dilogarithm (i.e., $a(X)$ for some $X \in A_2$) can be written as a sum of Euler dilogarithms (i.e., ℓi_2), products of logarithms and $n \cdot \frac{\pi^2}{6}$, $n \in \mathbb{Z}$.*

Proposition 3.2.3. $12 l_2(1) = 0$.

Proof. From the following picture of $l_2(1)$



one can see that $12(L;M) = (L;L) = 0$, where $L = (L_0, L_1, L_2)$ and $M = (M_0, M_1, M_2)$. □

Corollary 3.2.4. $(A_2(k)/P_2(k))_{\mathbb{Q}} \simeq B_2(k)_{\mathbb{Q}}$.

Proof. It is easy to see that for any half-square S, we have $2S \in P_2$. So the subgroup generated by half squares modulo P_2 is 2-torsion. □

See [Ünv11], for the analogue of this on k_2, that is $(A_2(k_2)/P_2(k_2))^{\circ} \simeq B_2(k_2)^{\circ}$.

3.3 A Construction of Mixed Tate Nori Motives

We will give a new construction of Aomoto polylogarithms based on [Nor04]. We will consider the motives coming from the following configurations. Fix $n \in \mathbb{N}^{>0}$. Let $B = \bigcup_{1 \le i \le m} B_i$, where all B_i are hyperplanes in B that meet $x_{i_1} = ... = x_{i_k} = 0$ properly for all $\{i_1, ..., i_k\} \subseteq \{1, ..., n\}$. We call such B a *nice divisor*. We will be interested in the motives of the form

$$H^n_{\text{Nori}}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m).$$

Proposition 3.3.1. *$H^n_{Nori}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m)$ is a mixed Tate motive with*

$$\operatorname{gr}^W_{2n} H^n_{Nori}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m) = \mathbf{1}(-n).$$

Proof. Let $M = H^n_{\text{Nori}}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m)$. Consider the double complex $F_q(D^p)$ for M as in the proof of theorem 2.6.2. On the first page of the spectral sequence we will have

$$\begin{array}{lllll}
c_{n,0} \cdot \mathbf{1}(-n) & & & & \\
c_{n-1,0} \cdot \mathbf{1}(-n+1) \to c_{n-1,1} \cdot \mathbf{1}(-n+1) & & & & \\
\vdots & \vdots & \vdots & & \\
c_{1,0} \cdot \mathbf{1}(-1) \longrightarrow & c_{1,1} \cdot \mathbf{1}(-1) \longrightarrow & \ldots \to c_{1,n-1} \cdot \mathbf{1}(-1) & & \\
c_{0,0} \cdot \mathbf{1}(0) \longrightarrow & c_{0,1} \cdot \mathbf{1}(0) \longrightarrow & \ldots \to c_{0,n-1} \cdot \mathbf{1}(0) \to & c_{0,n-1} \cdot \mathbf{1}(0) &
\end{array}$$

for some $c_{i,j}$. Hence M is mixed Tate.

Since the leftmost column comes from $\mathbb{G}^n_m$, we have $c_{i,0} = \binom{n}{i}$. In particular, $c_{n,0} = c_{0,0} = 1$. Hence $\text{gr}^W_{2n} M = c_{n,0} \cdot \mathbf{1}(-n) = \mathbf{1}(-n)$. □

One may also consider the exact sequence

$$0 \to n \cdot \mathbf{1}(-n+1) \to H^{n-1}_{\text{Nori}}(B \cap \mathbb{G}^n_m) \to H^n_{\text{Nori}}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m) \to \mathbf{1}(-n) \to 0.$$

Example 2.6.4 deals with $n = 1$ case. Assume $n > 1$. Then

$$\begin{aligned}
\text{gr}^W_{2n} H^n_{\text{Nori}}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m) &= \mathbf{1}(-n), \\
\text{gr}^W_0 H^n_{\text{Nori}}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m) &= \text{gr}^W_0 H^{n-1}_{\text{Nori}}(B \cap \mathbb{G}^n_m) = c \cdot \mathbf{1}(0),
\end{aligned}$$

where c is the number of cycles defined by B.

Corollary 3.3.2. *If $(L; B) \in A_n$, then $H^n_{Nori}(\mathbb{P}^n \setminus L, B \setminus (L \cap B))$ is a mixed Tate motive with*

$$\begin{aligned}
\text{gr}^W_{2n} H^n_{Nori}(\mathbb{P}^n \setminus L, B \setminus (L \cap B)) &= \mathbf{1}(-n), \\
\text{gr}^W_0 H^n_{Nori}(\mathbb{P}^n \setminus L, B \setminus (L \cap B)) &= \mathbf{1}(0).
\end{aligned}$$

Proof. Without loss of generality, we may assume that L is given by axis hyperplanes $z_i = 0$. Then, $M := H^n_{\text{Nori}}\left(\mathbb{P}^n \setminus L, B \setminus (L \cap B)\right) = H^n_{\text{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right)$ is mixed Tate. Since B is a simplex, $gr^W_0 M = \text{gr}^W_0 H^{n-1}_{\text{Nori}}\left(B \cap \mathbb{G}^n_m\right) = \mathbf{1}(0)$. □

Let

$$M = \bigoplus_{d \geq 0} M_d$$

where

$$M_d = \text{gr}^W_{2n-2d}(\varprojlim_B H^n_{\text{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right)) \otimes \mathbf{1}(n-d)$$

such that the limit is taken over all nice divisors B as in the beginning of the section. In particular,

$$M_0 = \mathbf{1}(0)$$

and

$$M_n = \text{gr}^W_0(\varprojlim_B H^n_{\text{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right)).$$

Let G be the Galois group of $\mathcal{MTM}_{\text{Nori},\mathbb{Q}}$. Then

$$1 \to U \to G \to \mathbb{G}_m \to 1$$

is split exact. Here, $U = \operatorname{Spec} R$, where $R = \bigoplus_{d \geq 0} R_d$ is a graded coalgebra and $\mathcal{MTM}_{\text{Nori},\mathbb{Q}}$ is equivalent to the category of graded R-comodules.

Conjecture 3.3.3 (Beilinson)**.** There is a natural isomorphism of graded Hopf algebras

$$A_{\mathbb{Q}} \xrightarrow{\sim} R.$$

Viewing M as a graded R-comodule, we have a linear map $\nu : M \to R \otimes M$. Let $\gamma_i : M \to M_i$ be the restriction map. Since $M_0 = \mathbf{1}(0)$ is realized as $\mathbb{Z}$, there is a natural map $\ell : M_0 \to \mathbb{Q}$. By composing

$$h : M \xrightarrow{\nu} R \otimes M \xrightarrow{\text{id}_R \otimes \gamma_0} R \otimes M_0 \xrightarrow{\text{id}_R \otimes \ell} R \otimes \mathbb{Q} \xrightarrow{\sim} R$$

we have a map $h : M \to R$ such that $h|_{M_0} = \ell$. This also gives

$$h|_{M_n} : M_n \to \bigoplus_{i+j=n} R_i \otimes M_j \to R_n \otimes M_0 \to R_n \otimes \mathbb{Q} \xrightarrow{\sim} R_n.$$

Let $G_n := S_n \ltimes \mathbb{G}_m^n$, where S_n is the symmetric group of order $n!$, and the action be given by $\sigma \cdot (a_1, \ldots, a_n) = (\sigma(a_1), \ldots, \sigma(a_n))$. Then G_n acts on $\mathbb{G}_m^n$ by

$$(\sigma \cdot a) \cdot x = (-1)^{|\sigma|} \sigma \cdot (ax)$$

for $\sigma \in S_n$, $a, x \in \mathbb{G}_m^n$. This action extends on M_n. Let

$$R'_n := H_0(G_n; M_n) = M_n / \langle gx - x \mid g \in G_n, x \in M_n \rangle.$$

Proposition 3.3.4. *$h|_{M_n}$ induces a map $\varphi_n : R'_n \to R_n$.*

Proof. By theorem 1.1.3, R_n is given by the framed objects. Coaction $M_n \to R_n \otimes M_n$ is given by frames

$$\mathbf{1}(0) \to \mathrm{gr}_0^W H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)$$

and it corresponds to the periods of $\mathrm{gr}_0^W H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)$. Without loss of generality, we may assmue that $\mathrm{gr}_0^W H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) = \mathbf{1}(0)$. Then, its period is of the form $\rho = \int_B \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n}$. But ρ is invariant under the action of both S_n and $\mathbb{G}_m^n$. So, the action of G_n respects the frames. Thus $h|_{M_n}(gx) = h|_{M_n}(x)$, for any $g \in G_n, x \in M_n$. □

Example 3.3.5. Let us consider the action of $\lambda \in G_1 = \mathbb{G}_m$ and motive

$$\mathrm{gr}_0^W H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{a, b\}).$$

The action is via multiplication by λ,

$$\mathrm{gr}_0^W H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{\lambda a, \lambda b\}) \to \mathrm{gr}_0^W H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{a, b\}).$$

Their periods are equal, $\log(\frac{\lambda b}{\lambda a}) = \log(\frac{b}{a})$. Hence

$$\begin{array}{ccc} \mathbf{1}(0) & \longrightarrow & \mathrm{gr}_0^W H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{a, b\}) \\ & \searrow & \uparrow \\ & & \mathrm{gr}_0^W H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{\lambda a, \lambda b\}) \end{array}$$

commutes.

Example 3.3.6. Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{G}_m^2$. Let us consider the action of $(12) \cdot \lambda \in G_2 = S_2 \ltimes \mathbb{G}_m^2$. Let B be a triangle. The action induces a map

$$\mathrm{gr}_0^W H^2_{\mathrm{Nori}}(\mathbb{G}_m^2, \lambda B \cap \mathbb{G}_m^2) \to \mathrm{gr}_0^W H^2_{\mathrm{Nori}}(\mathbb{G}_m^2, B \cap \mathbb{G}_m^2)$$

which respects the frames since

$$-\int_{\lambda B} \frac{dy}{\lambda_2 y} \wedge \frac{dx}{\lambda_1 x} = \int_B \frac{dx}{x} \wedge \frac{dy}{y}.$$

Let $R'_0 = \mathbb{Z}$ and $R' = \bigoplus_{n \geq 0} R'_n$. Tensor product of motives defines a multiplication $R'_{n'} \otimes R'_{n''} \to R'_n$. By the following lemma, the multiplications of R' and $A = \bigoplus_{n \geq 0} A_n$ are alike.

Lemma 3.3.7. *Assume $n' + n'' = n$. Let $(L'; B') \in A_{n'}$ and $(L''; B'') \in A_{n''}$. Then $(L'; B') \times (L''; B'') = \sum_i (L; B_i)$, for some $(L; B_i) \in A_n$. Assume that L, L', L'' are given by axis hyperplanes. Then,*

$$H^{n'}_{Nori}(\mathbb{G}_m^{n'}, B' \cap \mathbb{G}_m^{n'}) \otimes H^{n''}_{Nori}(\mathbb{G}_m^{n''}, B'' \cap \mathbb{G}_m^{n''}) = H^n_{Nori}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n),$$

where B is the nice divisor given by the union of simplices B_i.

Proof.

$$\begin{aligned}
&H^{n'}_{\mathrm{Nori}}(\mathbb{G}_m^{n'}, B' \cap \mathbb{G}_m^{n'}) \otimes H^{n''}_{\mathrm{Nori}}(\mathbb{G}_m^{n''}, B'' \cap \mathbb{G}_m^{n''}) \\
=&H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, \mathbb{G}_m^{n'} \times (B'' \cap \mathbb{G}_m^{n''}) \cup (B' \cap \mathbb{G}_m^{n'}) \times \mathbb{G}_m^{n''}) \\
=&H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, (\mathbb{G}_m^{n'} \times B'' \cup B' \times \mathbb{G}_m^{n''}) \cap \mathbb{G}_m^n) \\
=&H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n).
\end{aligned}$$

by the definition of multiplication in A. □

Theorem 3.3.8. *There is an isomorphism of graded algebras $\phi : R' \to A$.*

Proof. Let $n > 0$. Let $Z = (Z_0, \ldots, Z_n)$ be the n-simplex in $\mathbb{P}^n$ given by $Z_i : z_i = 0$. Define A'_n as the abelian group generated by (B) where B is an n-simplex in $\mathbb{P}^n$ such that (Z, B) is admissible, subject to the following relations:

1. If the hyperplanes of B are not in general position, then $(B) = 0$.

2. For every $\sigma \in S_n$,
$$(\sigma B) = (-1)^{|\sigma|}(B).$$

3. For every family of hyperplanes $B_0, ..., B_{n+1}$,
$$\sum(-1)^j(\hat{B}^j) = 0,$$
where $\hat{B}^j = (B_0, ..., \hat{B}_j, ..., B_{n+1})$.

4. For every $g \in \mathbb{G}_m^n$,
$$(gB) = (B),$$
where the action of $\mathbb{G}_m^n$ is as follows. For $g = (g_1, \ldots, g_n) \in \mathbb{G}_m^n$ and $p = (z_0 : z_1 : z_2 : \ldots : z_n) \in \mathbb{P}^n$, let $g \cdot p = (z_0 : g_1z_1 : g_2z_2 : \ldots : g_nz_n)$.

Define
$$\begin{aligned}\alpha : A'_n &\to A_n\\ (B) &\mapsto (Z; B).\end{aligned}$$

This map is well-defined since all 4 relations of A'_n hold for A_n, fixing the first index as Z. Define
$$\beta : A_n \to A'_n$$
in the following way. If $(L; M)$ is a generator of A_n, then $gL = Z$ for some $g \in \mathrm{PGL}_{n+1}$ and therefore $(L; M) = (Z; B)$, where $B = gM$. Set $\beta(L; M) := (B)$. We will show that this map is also well-defined. The first three relations of A_n hold for A'_n trivially. Let us show the last one also holds. Let V be the linearization of $\mathbb{P}^n$, and $\{e_0, e_1, \ldots, e_n\}$ be the standard basis for V. If $g \in \mathrm{PGL}_{n+1}$ is such that $Z = gZ$, then g maps e_i to g_ie_i for some $g_i \in \mathbb{G}_m$. Hence
$$g = \begin{pmatrix} g_0 & 0 & \ldots & 0 & \\ 0 & g_1 & 0 & \ldots & 0 \\ 0 & 0 & g_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & g_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & & \ldots & 0 & \\ 0 & g_1/g_0 & 0 & \ldots & 0 \\ 0 & 0 & g_2/g_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & g_n/g_0 \end{pmatrix} \in \mathbb{G}_m^n.$$

Thus β is well-defined as well. It is clear that α and β are inverses of each other. Hence they are isomorphisms.

Now we will write an isomorphism $R'_n \to A'_n$. We will consider the underlying $\mathbb{Z}$-modules of motives. We will work in the homological setting. By [Nor00, 4.4], the category of cohomological motives is isomorphic to the opposite category of homological motives. We denote by $H_n^{\text{Nori}}(X,Y)$ the corresponding object of $H^n_{\text{Nori}}(X,Y)$. Then,

$$M_n = \operatorname{gr}_0^W(\varinjlim_B H_n^{\text{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)),$$

such that the colimit is taken over all nice divisors B. We call a finite set of n-simplices *independent* if the relation (3) of the group A'_n is not satisfied for any $n+2$ choice of these simplices. A nice divisor B is called a *very nice divisor* if it is a union of independent simplices. If B is a nice divisor, by adding some hyperplanes we can find a very nice divisor $\tilde{B}$. Since $B \subseteq \tilde{B}$, there is a natural inclusion $\operatorname{gr}_0^W H^n_{\text{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \to \operatorname{gr}_0^W H^n_{\text{Nori}}(\mathbb{G}_m^n, \tilde{B} \cap \mathbb{G}_m^n)$. Hence M_n is the same as the colimit considering only very nice divisors.

Let B be a very nice divisor. If B is a simplex, define

$$\psi_B : \operatorname{gr}_0^W H^n_{\text{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) = \mathbf{1}(0) = \mathbb{Z} \to A'_n$$

as $\psi_B(1) = (B)$. If B is arbitrary, $\operatorname{gr}_0^W H_n^{\text{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)$ is the direct sum of the modules $\operatorname{gr}_0^W H_n^{\text{Nori}}(\mathbb{G}_m^n, B^i \cap \mathbb{G}_m^n) = \mathbb{Z}$ for finitely many independent simplices B^i. This defines a map $\psi_B : \operatorname{gr}_0^W H_n^{\text{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \to A'_n$. By relation (3) of A'_n, this is independent of the choice of B^i. Again by relation (3), all finite diagrams of such ψ_B commute. This extends a map

$$\psi : M_n \to A'_n.$$

If $(B) \in A'_n$ is a generator, then B is in the image of ψ_B. Hence ψ is surjective. We will show that $\ker\psi = \langle gx - x \mid g \in G_n, x \in M_n \rangle =: C_n$. By relation (2) and (4) of A'_n, we have $C_n \subseteq \ker\psi$. Let $x \in \ker\psi$. Then $x \in \operatorname{gr}_0^W H^n_{\text{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)$ for some very nice divisor B and $\psi_B(x) = 0$. Choose finitely many independent simplices B^i so that $\mathrm{gr}_0^W H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) = \bigoplus \mathrm{gr}_0^W H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B^i \cap \mathbb{G}_m^n)$. Call $x_i := x\big|_{B^i} \in \mathbb{Z}$. Since $\psi_B(x) = 0$, we have $\sum x_i(B^i) = \sum \psi_{B^i}(x_i) = \psi_B(x) = 0$. Since B^i are independent, without loss of generality we may assume that all relations between $x_i(B^i)$ comes from relations (2) and (4) of A'_n. Hence

$$x_{j_a} = -\sum x_{j_{a,b}}$$

and

$$B^{j_a} = g_{j_{a,b}} B^{j_{a,b}}$$

for some $g_{j_{a,b}} \in G_n$ such that $\{j_a\}_a \cup \{j_{a,b}\}_{a,b} = \{x_i\}_i$ and $\{j_a\}_a \cap \{j_{a,b}\}_{a,b} = \emptyset$. Therefore $x \in C_n$. Thus $\ker \psi = C_n$ and this gives the isomorphism

$$\psi' : R'_n = M_n/C_n \to A'_n.$$

We conclude that $\phi_n = \alpha \circ \psi' : R'_n \to A_n$ is an isomorphism, for each $n > 0$. For $n = 0$, we let $\phi_0 = \mathrm{id}_{\mathbb{Z}}$. By lemma 3.3.7, $\phi = \bigoplus_{n\geq 0} \phi_n$ respects multiplication. Thus ϕ is an isomorphism of graded algebras. □

Therefore, the comultiplication on A can be carried to R'. This makes R' a Hopf algebra. The comultiplication on A is defined as compatible with the comultiplication of Hodge-Tate structures. Hence it uses framings as the comultiplication of R. Hence, the comultiplications on R' and R are compatible and therefore $\varphi = \bigoplus \varphi_n$ is a morphism of graded Hopf algebras. We expect $\varphi_{\mathbb{Q}} = \varphi \otimes \mathbb{Q}$ to be an isomorphism.

Conjecture 3.3.9. $\varphi_{\mathbb{Q}} : R'_{\mathbb{Q}} \to R$ is an isomorphism.

This conjecture implies the Beilinson's conjecture (conjecture 3.3.3).

Let $\mathbb{B}$ be the full subdiagram of Good such that its vertices are of the form $(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n, n)$, where B is an n-simplex. We expect that R' is realized as the Hopf algebra of the diagram category of $\mathbb{B}$, i.e., $A(\mathbb{B}, H^*) = R'$ and therefore $\mathcal{C}(\mathbb{B}, H^*) = R' - \mathrm{Comod}$.

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